# Supermodularity and Preferences*

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**Abstract**

We uncover the complete ordinal implications of supermodularity on finite lattices under the assumption of weak monotonicity. In this environment, we show that supermodularity is ordinally equivalent to the notion of quasisupermodularity introduced by Milgrom and Shannon. We conclude that supermodularity is a weak property, in the sense that many preferences have a supermodular representation.

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# 1. Introduction

Economists have long regarded supermodularity as the formal expression of complementarity in preference; according to Samuelson (1947), the use of supermodularity as a notion of complementarities dates back to Fisher, Pareto and Edgeworth. In this paper, we characterize those weakly monotonic preferences on finite consumption sets that have a supermodular utility representation. We show that, in many ordinal economic models, supermodularity is a very weak assumption which is not testable with data on consumption expenditures.

Supermodularity is a cardinal property of a function defined on a lattice. It roughly states that a function has "increasing differences." For this reason, it is usually interpreted as modeling complementarities. For example, consider a consumer with a utility function over two goods. A seemingly natural notion of complementarity is that the two goods are complementary if the marginal utility of consuming one of the goods is increasing in the consumption of the other; for smooth functions, if the cross-partial derivatives are non-negative.

Because it is a cardinal property, a number of authors, including Allen (1934), Hicks and Allen (1934), Samuelson (1947), and Stigler (1950), rejected supermodularity as a notion of complementarities. They thought that, because it is not an ordinal notion, it would have no testable implications. While these authors were essentially right, their argument was incomplete. Their argument ran as follows: A supermodular function has positive cross-derivatives. Fix any given point. We may take an ordinal transformation of the function, preserving the preferences it represents, such that some cross-derivatives at this point become negative. While this is correct, it only demonstrates the lack of testable implications of supermodularity as *a local property.*[1] Supermodularity at any given point (in the form of nonnegative cross-derivatives) is not refutable. They believed that, as a consequence, supermodularity has no global ordinal or behavioral implica-

[1]Samuelson (1974) apparently understood this point as well (see p. 1271).

tions. In an influential and important paper, Milgrom and Shannon (1994) introduced quasisupermodularity, an ordinal implication of supermodularity. Any supermodular function generates a quasisupermodular order structure, proving that supermodularity has ordinal content.

Our main result characterizes the ordinal content of supermodularity under the additional assumptions of weak monotonicity and finiteness. Under these additional assumptions, it is equivalent to the more general notion of quasisupermodularity. We conclude that supermodularity can only have empirical implications in addition to quasisupermodularity if preferences are not weakly monotonic, or if an infinite number of preferences between objects can be observed. In addition, as all strictly monotonic preferences are quasisupermodular, we conclude that for strictly monotonic preferences, one cannot refute the hypothesis that a utility function is supermodular with a finite number of observations.

A primary implication of our results is that supermodularity is not testable with data on consumption expenditures. Thus, we rigorously confirm the intuition of Samuelson, Hicks, Allen and Stigler. We use a model related to Afriat (1967)'s, which shows that concavity is not testable with a finite collection of consumption data. We show that, in our model, any data which are generated by the preference maximization of a rational individual can also be generated by the maximization of a supermodular utility function.[2]

An additional application of our model is to the refutation of the ambiguity-aversion concept in the Choquet expected utility model, introduced by Schmeidler (1989). We find that, if we may only observe the amount of money that an individual would trade for a bet on a given event, and if this amount is monotonic with respect to set inclusion, then we cannot refute the hypothesis of ambiguity aversion.

Our results do not imply that supermodularity is a vacuous concept in economic theory. Supermodularity has proved a useful assumption in very different areas of economics. But the environments in which supermodular-

[2]Our model is different than Afriat's only in that we require consumption space to be finite. This seems a reasonable assumption if goods come in discrete units, for example.

ity is useful are either cardinal, such as convex transferable-utility games, or violate monotonicity, such as applications to games of strategic complementarities.

Under our maintained assumptions of finiteness and weak monotonicity, supermodularity has additional ordinal implications if imposed jointly with other conditions. Chipman (1977) showed that a differentiable, strongly concave and supermodular utility implies a normal demand. Quah (2007) shows that a concave and supermodular utility implies a class of monotone comparative statics; among other results, Quah generalizes Chipman's theorem.

We end with a note on the related literature. The seminal papers on supermodularity and lattice programming in economics are Topkis (1978), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994). The closest papers to ours are Kreps (1979) and Li Calzi (1990). Kreps proves a theorem that one can show is equivalent to our result (see Chambers and Echenique (2008)), but for the special case of a lattice of subsets. Li Calzi describes a class of functions which are strictly increasing transformations of supermodular functions. He is also the first to note a connection between monotonicity and supermodularity.

## 2. Definitions

Let $X$ be a set and $R$ be a binary relation on $X$. We define $xP_Ry$ if $xRy$ is true and $yRx$ is false. A **representation of** $R$ is a function $u : X \to \mathbb{R}$ for which $i$) for all $x, y \in X$, if $xRy$, then $u(x) \geq u(y)$, and $ii$) for all $x, y \in X$, if $xP_Ry$, then $u(x) > u(y)$.

A **partial order** on $X$ is a reflexive, transitive and antisymmetric binary relation on $X$. A **partially-ordered set** is a pair $(X, \preceq)$ where $X$ is a set and $\preceq$ is a partial order on $X$. A **lattice** is a partially-ordered set $(X, \preceq)$ such that for all $x, y \in X$, there exists a unique greatest lower bound $x \wedge y$ and a unique least upper bound $x \vee y$ according to $\preceq$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. We write $x \parallel y$ if neither $x \preceq y$ or $y \preceq x$. If $(X, \preceq)$ is a lattice, we say a subset $S$ of $X$ is larger than a subset $S'$ in the **strong set**

**order** if, for all $x \in S$ and all $y \in S'$, $x \vee y \in S$ and $x \wedge y \in S'$. We also use the notation $\partial S = \{x \in S : (\forall y \in X)(x \prec y \Rightarrow y \notin S)\}$ for the **order top boundary** of $S$.

Say that a function $u : X \to \mathbb{R}$ is **weakly increasing** if for all $x, y \in X$, $x \preceq y$ implies $u(x) \leq u(y)$. It is **strongly increasing** if for all $x, y \in X$, $x \preceq y$ and $x \neq y$ imply $u(x) < u(y)$. Say it is **weakly decreasing** if for all $x, y \in X$, $x \preceq y$ implies $u(y) \leq u(x)$. It is **weakly monotonic** if it is either weakly increasing or weakly decreasing.

A function $u : X \to \mathbb{R}$ is **quasisupermodular** if, for all $x, y \in X$, $u(x) \geq u(x \wedge y)$ implies $u(x \vee y) \geq u(y)$ and $u(x) > u(x \wedge y)$ implies $u(x \vee y) > u(y)$. A function $u : X \to \mathbb{R}$ is **supermodular** if, for all $x, y \in X$, $u(x \vee y) + u(x \wedge y) \geq u(x) + u(y)$.

## 3. Supermodular Representation

We discuss conditions under which $R$ has a supermodular representation. We first present the equivalence of supermodularity and quasisupermodularity for weakly monotonic functions on finite lattices.

Let $(X, \preceq)$ be a finite lattice. Our primary result is the following.[3,4]

Theorem 1: *A binary relation on $X$ has a weakly increasing and quasisupermodular representation if and only if it has a weakly increasing and supermodular representation.*

Proof: Any weakly increasing and supermodular representation is also weakly increasing and quasisupermodular. Thus, suppose that $u : X \to \mathbb{R}$ is a weakly increasing and quasisupermodular representation. We will show that there exists some $f : u(X) \to \mathbb{R}$ which is strictly increasing, for which $f \circ u$

[3] Our original proof of this result was significantly longer and non-constructive. It relied on an integer version of the Theorem of the Alternative. The present proof follows the suggestions of John Quah and Eran Shmaya.

[4] The theorem can be slightly generalized to the case in which $(X, \preceq)$ is an arbitrary lattice and the binary relation under consideration has a finite number of equivalence classes. We thank Eran Shmaya for this observation.

is supermodular. As $X$ is finite, $u(X)$ is also finite. Label the elements of $u(X)$ as $\{u_1, ..., u_N\}$, where $u_1 < u_2 < ... < u_N$. Define $f : u(X) \to \mathbb{R}$ as $f(u_j) \equiv 2^{j-1}$. Note that $f \circ u$ is weakly increasing. Moreover, it is also quasisupermodular. Label $g \equiv f \circ u$. We will establish that $g$ is supermodular. To this end, let $x, y \in X$. If $x \succeq y$ or $y \succeq x$, then it is obvious that $g(x) + g(y) \leq g(x \wedge y) + g(x \vee y)$. Thus, suppose that $x \parallel y$. By monotonicity, we know that $g(x) \leq g(x \vee y)$ and $g(y) \leq g(x \vee y)$. We claim that we may without loss of generality assume that these inequalities are strict. To see this, suppose that $g(x) = g(x \vee y)$. By monotonicity, $g(y) \geq g(x \wedge y)$. By quasisupermodularity, $g(y) = g(x \wedge y)$. Consequently, $g(x) + g(y) = g(x \vee y) + g(x \wedge y)$. The proof for the case $g(y) = g(x \vee y)$ is symmetric.

Hence, $g(x) < g(x \vee y)$ and $g(y) < g(x \vee y)$. Let $k \in \{1, ..., N\}$ be such that $u(x \vee y) = u_k$. Then $u(x) \leq u_{k-1}$ and $u(y) \leq u_{k-1}$. Hence

$$g(x \vee y) + g(x \wedge y) \geq 2^{k-1} = 2^{k-2} + 2^{k-2} \geq g(x) + g(y).$$

Hence $g$ is supermodular. $\square$

Remark 1: In Theorem 1, the term "weakly increasing" can be replaced by "weakly decreasing" (this is the version shown by Kreps (1979) for the case of a lattice of subsets). The argument is as follows. Suppose that $u$ is a weakly decreasing and quasisupermodular function on the lattice $(X, \preceq)$. For all $x, y \in X$, define $x \preceq' y$ if $y \preceq x$. Then $(X, \preceq')$ is also a lattice; moreover, $x (\wedge') y = x \vee y$ and $x (\vee') y = x \wedge y$. Further, $u$ is weakly increasing on $(X, \preceq')$. We show that $u$ is also quasisupermodular on $(X, \preceq')$. To see this, note that $u(x) \geq u(x (\wedge') y)$ implies $u(x) \geq u(x \vee y)$. Therefore, as $u$ is quasisupermodular on $(X, \preceq)$, it follows that $u(x \wedge y) \geq u(y)$ (otherwise, if $u(y) > u(x \wedge y)$, quasisupermodularity would imply that $u(x \vee y) > u(x)$). Hence $u(x (\vee') y) \geq u(y)$. If $u(x) > u(x (\wedge') y)$, then $u(x) > u(x \vee y)$. By quasisupermodularity of $u$ on $(X, \preceq)$, this implies that $u(x \wedge y) > u(y)$ (otherwise, if $u(y) \geq u(x \wedge y)$, then by quasisupermodularity $u(x \vee y) \geq u(x)$). Hence $u(x (\vee') y) > u(y)$. Thus $u$ is quasisupermodular on $(X, \preceq')$.

Hence, there exists a strictly increasing $f : u(X) \to \mathbb{R}$ such that $f \circ u$ is supermodular on $(X, \preceq')$. By the supermodularity of $f \circ u$ on $(X, \preceq')$, for all $x, y \in x$, $(f \circ u)(x) + (f \circ u)(y) \leq (f \circ u)(x (\wedge') y) + (f \circ u)(x (\vee') y) = (f \circ u)(x \wedge y) + (f \circ u)(x \vee y)$, establishing the supermodularity of $f \circ u$ on $(X, \preceq)$.

Li Calzi (1990) provides a related result whose proof also relies on the composition of a function with an exponential function.

The following examples demonstrate that the equivalence described in the theorem may fail to hold under weaker conditions.

To see that weak monotonicity alone is not sufficient for supermodularity, consider the following example.

EXAMPLE 2: Let $X = \{1, 2\}^2$ with the usual ordering. Let $R$ be representable by the function $u : X \to \mathbb{R}$ for which $u((1, 1)) = 0$, and $u((1, 2)) = u((2, 1)) = u((2, 2)) = 1$. Note that $R$ cannot be represented by a supermodular function (any such function $v$ would require $v((1, 2)) = v((2, 1)) = v((2, 2)) > v((1, 1))$, so that $v((1, 1)) + v((2, 2)) < v((2, 1)) + v((1, 2))$. Nevertheless, $R$ is weakly monotonic. $R$ cannot be represented by a quasisupermodular function, as $u((1, 2)) > u((1, 1))$, yet $u((2, 1)) \geq u((2, 2))$.

To see that monotonicity is not necessary for a supermodular representation, consider the following example.

EXAMPLE 3: Let $X = \{1, 2\}^2$ with the usual ordering. Let $R$ be representable by the function $u : X \to \mathbb{R}$ for which $u((1, 1)) = 0$, $u((1, 2)) = -1$, $u((2, 1)) = 2$, and $u((2, 2)) = 1.5$. Note that $u$ is (strictly) supermodular. However, note that $R$ is not monotonic.

The preceding discussion suggests one may want to find all binary relations with a supermodular representation. We present the somewhat complicated answer as Theorem 10 in the appendix.

### 3.1. *Strong Monotonicity, Monotone Comparative Statics and Supermodularity*

Theorem 1 has an important immediate corollary. The corollary follows because any strongly increasing function is quasisupermodular.

> COROLLARY 4: *Let $(X, \preceq)$ be a finite lattice. If a binary relation has a strongly increasing representation, then it has a supermodular representation.*

PROOF: Let $R$ be a binary relation with a strongly increasing representation $u$. We claim that $u$ is quasisupermodular. Let $x, y \in X$. First, $x \succeq x \wedge y$ and $x \vee y \succeq y$, so $u(x) \geq u(x \wedge y)$ and $u(x \vee y) \geq u(y)$ hold. Second, if $u(x) > u(x \wedge y)$, then $x \succeq x \wedge y$ and $x \neq x \wedge y$. Hence, $x \vee y \succeq y$, and $x \neq x \wedge y$ implies $x \vee y \neq y$, so $u(x \vee y) > u(y)$. By Theorem 1, $R$ has a supermodular representation. $\square$

Corollary 4 states that any consumer with strongly monotonic preferences can be viewed as maximizing a supermodular utility function. This observation implies that supermodularity, the property of utility which has traditionally been viewed as implying that goods are complementary, is vacuous in a strongly monotonic environment.

As shown by Milgrom and Shannon (1994), quasisupermodular functions are the only objective functions whose comparative statics are monotone for all monotone changes in the constraint set. A combination of Theorem 1 with their results leads to an alternative characterization of supermodular functions. We present some definitions and then reinterpret Milgrom and Shannon's monotonicity theorem as our Corollary 5.

Let $u : X \to \mathbb{R}$. Define the correspondence $M^u : 2^X \backslash \varnothing \rightrightarrows 2^X \backslash \varnothing$ as $M^u(S) = \{x \in S : u(y) \leq u(x) \text{ for all } y \in S\}$. Say that $M^u$ exhibits **non-satiation** if for all $S$, $\partial S \cap M^u(S) \neq \emptyset$. Say that $M^u$ is weakly increasing if $M^u(S)$ is smaller than $M^u(S')$ in the strong set order whenever $S$ is smaller than $S'$ in the strong set order.

Corollary 5: *There exists a strictly increasing* $f : \mathbb{R} \to \mathbb{R}$ *such that the function* $f \circ u : X \to \mathbb{R}$ *is weakly increasing and supermodular if and only if* $M^u$ *is weakly increasing and exhibits non-satiation.*

The proof of Corollary 5 is immediate from Milgrom and Shannon's Monotonicity Theorem (Theorem 4 in Milgrom and Shannon (1994)) and our Theorem 1.

## 4. Application 1: Non-Refutability with Consumption Data

Afriat (1967) studies data on consumption choices at different prices. He shows that data can arise from a rational consumer if and only if one can model the consumer using a concave utility function. Concavity of utility is thus not refutable with data on consumption choices. Assuming a finite consumption space, we show that data can arise from a rational consumer if and only if one can model the consumer using a supermodular utility.

Let $X \subseteq \mathbb{R}^n_+$ be a finite lattice. For all $k = 1, ..., K$, let $S^k \in 2^X \setminus \{\emptyset\}$ and let $x^k \in S^k$. A pair $(x^k, S^k)$ is a consumption bundle $x^k$ demanded by a consumer when the feasible set is $S^k$. This is a natural framework in which to investigate the empirical content of supermodularity in consumer theory. The model is close to Matzkin's (1991) and Forges and Minelli's (2006) generalization of Afriat's model.

We assume data satisfy the following three conditions. The first is a classical "non-satiation" assumption. The second is a "free disposability" assumption. The last requirement is a variant of the weak axiom of revealed preference. It ensures that the revealed-preference relation is asymmetric.[5]

1. For all $k \in \{1, ..., K\}$, $x^k \in \partial S^k$.

[5] Since we are studying the issue of rationalizability, assuming the weak axiom is vacuous because it is necessary for the existence of a rationalization. We make the assumption for expositional reasons, to work with the same notion of strict preference as in the rest of the paper.

2. For all $k \in \{1, ..., K\}$, if $x \in S^k$ and $y \leq x$ then $y \in S^k$.

3. For all $k, k' \in \{1, ..., K\}$, if $x^{k'} \in S^k$ and $x^{k'} \neq x^k$, then $x^k \notin S^{k'}$.

Define $R$ on $X$ by $xRy$ if there exists $k \in \{1, ..., K\}$ such that $x = x^k$ and $y \in S^k$. Note that $R$ is the standard revealed-preference relation. By 3, for all $x, y \in X$, if $xRy$ and $x \neq y$, then $xP_Ry$.

Say that $u : X \to \mathbb{R}$ **rationalizes** the data $\left\{(x^k, S^k)\right\}_{k=1}^K$ if it represents $R$. In other words, $u$ rationalizes the data if a consumer maximizing utility $u$ in $S^k$ chooses $x^k$. The function $u$ is called a rationalization of the data.

> PROPOSITION 6: *Assume that $\left\{\left(x^k, S^k\right)\right\}_{k=1}^K$ satisfies 1,2, and 3. Then $\left\{(x^k, S^k)\right\}_{k=1}^K$ is rationalizable if and only if it is rationalized by a supermodular function.*

PROOF: We write $xRy$ for $(x, y) \in R$ and $xR^\tau y$ if there is a sequence

$$x = x_1, x_2, \ldots, x_K = y$$

with $K > 1$ and $x_k R x_{k+1}$, for $k = 1, \ldots K - 1$. Let $B$ be the binary relation defined as $xBy$ if $xRy$ or $x \geq y$. It is easily verified that $xP_By$ if $xP_Ry$ or $x > y$.

To prove the proposition, we establish that $R$ has a supermodular representation. First, we show that any representation of $B$ is also a representation of $R$. Second, we show that if $R$ has a representation, then $B$ has a representation. As a consequence, if $\left\{(x^k, S^k)\right\}_{k=1}^K$ is rationalizable, $B$ has a representation. Any representation of $B$ is strongly monotone and is also a representation of $R$. Corollary 4 then implies the existence of a supermodular rationalization.

We show that any representation of $B$ is also a representation of $R$: let $u$ represent $B$. First let $xRy$. Then $xBy$, and hence $u(x) \geq u(y)$. Second, let $xP_Ry$. Then $xBy$; we prove that in fact $xP_By$. If $yBx$, then $y \geq x$ or $yRx$, but the latter would violate Property (3) of the data. Hence $yBx$ implies $y \geq x$. But as $xP_Ry$, $x \neq y$, hence $y > x$. Hence there exists $k \in \{1, ..., K\}$

for which $x = x^k$ and $y \in S^k$. But this implies that $x^k \notin \partial S^k$, a contradiction of Property (1). Hence $x P_B y$ and thus $u(x) > u(y)$.

We now reproduce, as Lemma 7 and without proof, the standard result on when $R$ has a representation (see e.g. Richter (1966)).

> LEMMA 7: *There is a representation of $R$ if and only if $xR^\tau y$ implies that $yP_R x$ is false.*

We show that, if $R$ has a representation, then so does $B$. Suppose that $B$ has no representation. Then there exist $x, y \in X$ for which $xB^\tau y$ and $yP_B x$. Suppose that $x \geq y$. Then $yP_B x$ implies that there exists $k$ for which $y = x^k$. But $y \in \partial S^k$, so $x \geq y$ implies $x = y$, contradicting the fact that $yP_B x$. Therefore, $x \ngeq y$. Hence, $xB^\tau y$ implies that there exists $\{x_1, \ldots x_L\} \subseteq X$ for which $xBx_1B \ldots Bx_LBy$, where at least one $B$ corresponds to $R$ and not $\geq$.

We claim that there exists $x' \neq y$ for which $x \geq x'$ and $x'R^\tau y$. To see this, note that for any collection of data $\{z_1, ..., z_m\} \subseteq X$ for which $z_1Rz_2 \geq z_3 \geq ... \geq z_m$, it follows by the fact that $z_2 \geq z_m$ and Property 2 of the data that $z_1Rz_m$. From this fact, we establish that there is $x' \neq y$ with $x \geq x'$ and $x'R^\tau y$.

As $yP_B x$, either $yP_R x$ or $y > x$. If $yP_R x$, then $yP_R x'$ by $x \geq x'$ and Properties 1 and 2 of the data. In this case, we may conclude by Lemma 7 that $R$ has no representation. If instead $y > x$, then we have $x'R^\tau y$ and $y > x'$. Then, $x'Ry$ contradicts Property 1 of the data, so there exists $x'' \in X$, $x'' \neq x'$ for which $x'R^\tau x''Ry$. Now, $y \geq x'$ and Property 2 of the data imply that $x''Rx'$. In fact, by Property 3 and $x'' \neq x'$, $x''P_R x'$. So Lemma 7 implies that $R$ does not have a representation. $\square$

Our model is related to Afriat's in the following way. Afriat assumes that $X = \mathbb{R}^n_+$ (a case our Proposition 6 does not cover). His data consists of pairs $\{(x^k, p^k)\}_{k=1}^K$ for which $x^k \in X \subseteq \mathbb{R}^n_+$ and $p^k \in \mathbb{R}^n_{++}$, for all $k$. Each pair is an observed consumption choice $x^k$ at prices $p^k$. Afriat's data obtains when

$$S^k = \{x : p^k \cdot x \leq p^k \cdot x^k\}.$$

Note that the weak axiom, Property 3 of the data, is implicit in Afriat's results. If we do not assume Property 3 we would need to distinguish between representability of $R$ and rationalization; the result in Proposition 6 relating rationalizability with a supermodular rationalization, continues to hold.

Our model allows us to accommodate non-linear budgets sets. Forges and Minelli (2006) investigate a similar model of non-linear budget sets, and establish that concavity has stronger implications than rationality alone. Thus the non-refutability of supermodularity is more robust than that of concavity, since it continues to hold with general budget sets. Matzkin (1991) first discussed non-linear budget sets to incorporate situations where consumers have monopsony power, or where the consumer is a social planner facing an economy's production possibility set.

Two remarks are in order. First, under an additional restriction on Afriat's data, rationalizability implies that there is a smooth, strongly monotonic rationalization (Chiappori and Rochet, 1987). Corollary 20 in Li Calzi (1990) then implies the existence of a supermodular rationalization. So one can use existing results to prove a version of Proposition 6 for the Afriat data under Chiappori and Rochet's assumptions.

The second remark refers to concave rationalizations. Afriat shows that data are rationalizable if and only if they are rationalizable by a concave utility. Proposition 6 says that data are rationalizable if and only if they are rationalizable by a supermodular utility. One might conjecture that any rationalizable data con be rationalized by a function that is both concave and supermodular. This turns out to be false. Supermodularity and concavity jointly imply that demand is normal. So, while concavity and supermodularity have no testable implications as individual assumptions, they are refutable as joint assumptions.

That supermodularity and concavity imply normal demand is shown in Quah (2007); the earlier result of Chipman (1977) requires additional smoothness assumptions on utility. Quah's result does not apply to functions that are supermodular on a finite domain. We present a very simple adaptation of Quah's argument in Example 8.

EXAMPLE 8: Consider the data $(x^k, p^k)_{k=1}^2$, where $p^1 = (2, 1.1)$, $x^1 = (1, 2)$, $p^2 = (2.1, 1)$, and $x^2 = (2, 1)$. This collection of data is rationalizable (as it satisfies Afriat's condition) and thus has both a concave and a supermodular rationalization. We show that it has no concave and supermodular rationalization.

Let $C \subseteq \mathbb{R}^2$ be convex and $X \subseteq C$ be a sublattice such that

$$\{(1, 2), (2, 1), (3/2, 1), (3/2, 2), (2, 2)\} \subseteq X.$$

Suppose that $u : C \to \mathbb{R}$ is concave and that $u|_X$ is supermodular. We shall prove that $u$ cannot rationalize the data.

We first note that $p^1 \cdot (3/2, 1) < p^1 \cdot (1, 2)$ so that we need $u(3/2, 1) < u(1, 2)$ for $u$ to rationalize the data. We then prove that $u(2, 1) < u(3/2, 2)$, which is inconsistent with $u$ rationalizing the data, as $p^2 \cdot (3/2, 2) < p^2 \cdot (2, 1)$. Start from $u(3/2, 2) - u(2, 1) = u(3/2, 2) - u(2, 2) + u(2, 2) - u(2, 1)$. Then,

$$\begin{aligned} u(3/2, 2) - u(2, 2) &= u\left((2, 2) - (1/2)(1, 0)\right) - u(2, 2) \\ &\geq u\left((2, 2) - (1/2)(1, 0) - (1/2)(1, 0)\right) \\ &\quad -u\left((2, 2) - (1/2)(1, 0)\right) \\ &= u(1, 2) - u(3/2, 2); \end{aligned}$$

the inequality above follows from concavity (Quah, 2007). Supermodularity on $X$ implies that $u(2, 2) - u(2, 1) \geq u(3/2, 2) - u(3/2, 1)$. Hence,

$$\begin{aligned} u(3/2, 2) - u(2, 1) &\geq u(1, 2) - u(3/2, 2) + u(3/2, 2) - u(3/2, 1) \\ &= u(1, 2) - u(3/2, 1). \end{aligned}$$

This implies that $u$ cannot rationalize the data because $u(3/2, 1) < u(1, 2)$ implies $u(2, 1) < u(3/2, 2)$.

Quah's result on normal demand is not directly applicable because the domain on which $u$ is supermodular is finite, and because prices and expenditure both change between observations $k = 1$ and $k = 2$. However, a straightforward modification of Quah's arguments gives the result.

## 5. Application 2: Uncertainty aversion and the Choquet expected utility model.

We now turn to a model of decision under uncertainty where supermodularity models uncertainty aversion.

An individual faces risk when probabilities are exogenously specified. If she is not given these probabilities, she faces uncertainty. When the events are not given probabilities, there is no reason to suspect that the individual will assign probabilities to them. We study a model introduced by Schmeidler (1989); in this model, the individual in fact need not assign probabilities to events, but assigns some measure of likelihood to them. This measure is called a capacity. Supermodularity of the capacity in this model is interpreted as uncertainty aversion.

While it is well-known that Schmeidler's notion of uncertainty aversion may not place restrictions on betting behavior, we show that the extent to which this is the case may be greater than previously realized.

We briefly describe Schmeidler's model and explain the implications of our results.

Let $\Omega$ be a finite set of possible states of the world and let $Y$ be a set of possible outcomes. The set of (Anscombe and Aumann, 1963) **acts** is the set of functions $f : \Omega \to \Delta(Y)$. Denote the set of acts by $\mathcal{F}$. A **capacity** is a function $\nu : 2^{\Omega} \to \mathbb{R}$ for which $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$, and $A \subseteq B$ implies $\nu(A) \leq \nu(B)$. A capacity is **supermodular** if it is supermodular when $2^{\Omega}$ is endowed with the set-inclusion order.

A binary relation $R$ over $\mathcal{F}$ conforms to the **Choquet expected utility model** if there exists some $u : \Delta(Y) \to \mathbb{R}$ conforming to the von Neumann-Morgenstern axioms and a capacity $\nu$ on $\Omega$ for which the function $U : \mathcal{F} \to \mathbb{R}$

represents $R$, where

$$U(f) \equiv \int_{\Omega} u(f(\omega))\, d\nu(\omega)\,;^{6} \tag{0}$$

Schmeidler (1989) axiomatizes those $R$ conforming to the Choquet expected utility model. A binary relation $R$ which conforms to the Choquet expected utility model exhibits **Schmeidler uncertainty aversion** if and only if $\nu$ is supermodular.

For a given binary relation $R$ over $\mathcal{F}$ conforming to the Choquet expected utility model, define the likelihood relation $R^*$ over $2^{\Omega}$ by $ER^*F$ if there exist $x, y \in X$ for which $xP_Ry$[7] and

$$\left[\begin{array}{c} x \text{ if } \omega \in E \\ y \text{ if } \omega \notin E \end{array}\right] R \left[\begin{array}{c} x \text{ if } \omega \in F \\ y \text{ if } \omega \notin F \end{array}\right].$$

The likelihood relation reflects a "willingness to bet." If $ER^*F$, then the individual prefers to place stakes on $E$ as opposed to $F$. For the Choquet expected utility model, this relation is complete. We will write the asymmetric part of $R^*$ by $P^*$ and the symmetric part by $I^*$. The following is an immediate corollary to Theorem 1, using the fact that all likelihood relations are weakly increasing.

> PROPOSITION 9: *Suppose that $R$ conforms to the Choquet expected utility model. Then the likelihood relation $R^*$ is incompatible with Schmeidler uncertainty aversion if and only if there exist events $A, B, C \subseteq \Omega$ for which $A \subseteq B$ and $B \cap C = \varnothing$ for which $(A \cup C)\, P^* A$ and $(B \cup C)\, I^* B$.*

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[6] The Choquet integral with respect to $\nu$ is defined as:

$$\begin{aligned} & \int_{\Omega} g(\omega)\, d\nu(\omega) \\ = \; & \int_{0}^{+\infty} \nu(\{\omega : g(\omega) > t\})\, dt + \int_{-\infty}^{0} [\nu(\{\omega : g(\omega) > t\}) - 1]\, dt \end{aligned}$$

[7] Here we are abusing notation by identifying a constant act with the value that constant act takes.

# A. A Characterization of Supermodular Representation

We obtain a characterization of the relations with a supermodular representation from an application of the Theorem of the Alternative. Similar results for concave representations are Richter and Wong (2004) and Kannai (2005).

Let $(X, \preceq)$ be a finite lattice.

Theorem 10: *A binary relation $R$ on $X$ has a supermodular representation if and only if, for all $N, K \in \mathbb{N}$, for all $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$, $\{z_l\}_{l=1}^K$, $\{w_l\}_{l=1}^K \subseteq X$ for which for all $l = 1, ..., K-1$, $z_l R w_l$ and for which*

$$\sum_{i=1}^{N} \left(1_{x_i \vee y_i} + 1_{x_i \wedge y_i}\right) + \sum_{l=1}^{K} 1_{z_l} = \sum_{i=1}^{N} \left(1_{x_i} + 1_{y_i}\right) + \sum_{l=1}^{K} 1_{w_l},$$

*$z_K P_R w_K$ does not hold.*

Proof: The existence of a supermodular representation is equivalent to the existence of a vector $u \in \mathbb{R}^X$ for which $i)$ for all $x, y \in X$ for which $x \parallel y$, $(1_{x \vee y} + 1_{x \wedge y} - 1_x - 1_y) \cdot u \geq 0$, and $ii)$ for all $x, y \in X$ for which $x P_R y$, $(1_x - 1_y) \cdot u > 0$. By the integer version of the Theorem of the Alternative (Aumann, 1964; Fishburn, 1970), such a vector fails to exist if and only if for all $x, y \in X$ for which $x \parallel y$, there exists some $n_{\{x,y\}} \in \mathbb{Z}_+$ and for all $x, y \in X$ for which $xRy$, there exists some $n_{(x,y)} \in \mathbb{Z}_+$, and there exists at least one $n_{(x,y)} > 0$ for which $x P_R y$, such that

$$\sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} \left(1_{x \vee y} + 1_{x \wedge y} - 1_x - 1_y\right) + \sum_{\{(x,y):xRy\}} n_{(x,y)} \left(1_x - 1_y\right) = 0.$$

Separating terms, we obtain

$$\begin{aligned} &\sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} \left(1_{x \vee y} + 1_{x \wedge y}\right) + \sum_{\{(x,y):xRy\}} n_{(x,y)} 1_x \\ = &\sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} \left(1_x + 1_y\right) + \sum_{\{(x,y):xRy\}} n_{(x,y)} 1_y. \end{aligned}$$

It is easy to see that this is equivalent to the existence of $N, K \in \mathbb{N}$, $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$, $\{z_l\}_{l=1}^K$, $\{w_l\}_{l=1}^K \subseteq X$ such that for all $l = 1, ..., K-1$, $z_l R w_l$ and for which

$$\sum_{i=1}^N (1_{x_i \vee y_i} + 1_{x_i \wedge y_i}) + \sum_{l=1}^K 1_{z_l} = \sum_{i=1}^N (1_{x_i} + 1_{y_i}) + \sum_{l=1}^K 1_{w_l},$$

and $z_K P_R w_K$. □

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