Solving Large-Scale Linear Circuit Problems via Convex Optimization

Javad Lavaei, Aydin Babakhani, Ali Hajimiri and John C. Doyle

Abstract— **A broad class of problems in circuits, electromagnetics, and optics can be expressed as finding some parameters of a linear system with a specific type. This paper is concerned with studying this type of circuit using the available control techniques. It is shown that the underlying problem can be recast as a rank minimization problem that is NP-hard in general. In order to circumvent this difficulty, the circuit problem is slightly modified so that the resulting optimization becomes convex. This interesting result is achieved at the cost of complicating the structure of the circuit, which introduces a trade-off between the design simplicity and the implementation complexity. When it is strictly required to solve the original circuit problem, the elegant structure of the proposed rank minimization problem allows to employ a celebrated heuristic method to solve it efficiently.**

I. INTRODUCTION

A vast majority of problems in circuits, electromagnetics, and optics can be regarded as the analysis and synthesis of linear systems in the frequency domain. These systems, in the circuit theory, consist of passive elements including resistors, inductors, capacitors, ideal transformers, and ideal gyrators [1]. Since the seminal work [2], there has been remarkable progress in characterizing such passive (dissipative) systems using the concept of positive real functions. This notion plays a vital role not only in circuit design but also in various control problems [1], [3], [4].

The application of control theory in circuit and communication areas evidently goes beyond the passivity concept. Indeed, the emerging optimization tools developed by control theorists, such as linear matrix inequalities (LMIs) [5] and sum-of-squares (SOS) [6], have been successfully applied to a number of fundamental problems in these fields. For instance, the recent paper [7] proposes an LMI optimization to check whether a given multi-port network can be realized using a pre-specified set of linear time-invariant components (namely an inductor and small-signal model of a transistor). Moreover, the work [8] formulates the pattern synthesis of large arrays with bound constraints on the sidelobe and mainlobe levels as a semidefinite program.

It is well-known that a broad class of problems in circuits, electromagnetics, and optics can be formulated as an optimization over the parameters of a multi-port passive network which is obtained, for instance, via an electromagnetic (EM) simulation. As an example, it is shown in [9] that a strikingly efficient and practical way to deal with certain complex antenna problems is to extract a circuit model and then search for appropriate values of its parameters. The circuit model proposed in [9] is indeed a simple, general model which could be considered the abstract model of different types of problems. A question arises as to whether there exists a systematic method to study such circuit problems. This paper basically aims to address this question using the available techniques developed in the control theory, especially the LMI and passivity concepts.

Motivated by the work [9], a general circuit problem is considered in this paper, which requires finding a set of parameters to satisfy some prescribed design specifications. It is shown that this problem amounts to solving a simple rank-minimization problem, which is known to be NP-hard in general. Afterwards, the circuit problem is slightly modified to make the resulting optimization problem convex. The convexity proof provided here mainly relies on the elegant properties of passive networks and the power of LMI techniques. The modification made in the circuit problem does not noticeably alter the original circuit or electromagnetic problem, but makes its implementation harder in practice. For this reason, the heuristic method proposed in [10] (and further studied in [11]) is subsequently applied to the obtained rank-minimization problem. It is observed that this heuristic method works satisfactorily, to a great extent.

The paper is organized as follows. In section II, an in-depth discussion is provided to outline the deficiencies of the current techniques used to study an electromagnetic problem. Following this motivation, the main results are developed in Section III. Simulation results are then given in Section IV to demonstrate the efficacy of this work. Finally, some concluding remarks are drawn in Section V

II. MOTIVATION AND PROBLEM FORMULATION

This section first highlights the advantages of solving electromagnetic problems based on their circuit models, and then formalizes the objective of the present work.

Indeed, numerical methods and efficient optimization techniques, enabled by increasing computational power, have been markedly instrumental in advancing the field of modern electrodynamics. The progress in this field which was limited to the development of analytical models for antenna characteristics such as pattern, efficiency, and impedance, has been greatly influenced by novel numerical techniques in time or frequency domains. Frequency domain techniques such as finite element method [12] and method of moments [13], as well as time domain algorithms such as finite difference

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technique [14], have been extensively used in designing electromagnetic structures. These numerical methods combined with optimization techniques such as genetic algorithm [15] and particle swarm optimization [16] provide a valuable, but inefficient, tool in designing large-scale electromagnetic structures where thousands of passive elements are involved. More precisely, the available numerical techniques iteratively search for a sub-optimal solution. Since a new time-consuming EM simulation needs to be run at each iteration, this approach could be really prohibitive, due to the exponential number of iterations. In the recent paper [9], this crucial issue is partially resolved by introducing a novel method, which requires performing the EM simulation only once to extract the scattering parameters of the system. Having obtained this circuit model, the elctromagnetic problem reduces to solving a non-iterative optimization problem over the parameters of the circuit.

The circuit model of the above-mentioned problem is depicted in Figure 1, in which there are n switches at the output ports. The goal is to find a collection of these switches whose connection causes a number of linear constraints on the output voltages and the input impedance to be satisfied. The continuous counterpart of this discrete problem supplants the switches in the output ports with varactors and, therefore, transforms the objective to finding the values of the varactors. Roughly speaking, every problem governed by Maxwells differential equations that requires finding optimal values of the termination impedances can be converted to a similar discrete or continuous circuit optimization.

III. Main results

A. Discrete circuit optimization

To analyze the circuit given in Figure 1, define the currents $i_1, i_2, ..., i_{n+1}$ and the voltages $v_1, v_2, ..., v_{n+1}$ as shown in the figure. For simplicity, introduce the shorthand notations:

$$\begin{aligned} \tilde{\mathbf{i}} &= \begin{bmatrix} i_1 & i_2 & \cdots & i_n \end{bmatrix} \\ \tilde{\mathbf{v}} &= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \end{aligned} \tag{1}$$

One can write two sets of equations as follows:

$$\begin{bmatrix} \tilde{\mathbf{i}} & i_{n+1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{v}} & v_{n+1} \end{bmatrix} Y_s \tag{2a}$$

$$i_{n+1} = y_{\text{in}} \cdot v_{n+1} \tag{2b}$$

where:

- Y_s is the given Y-parameter matrix of a reciprocal, strictly passive network at a specific frequency ω_0.
- v_{n+1} is equal to the input voltage v_{in}.
- y_{in} is the input admittance.

It is worth noting that Y_s is a complex-valued matrix whose real and imaginary parts are both symmetric. With no loss of generality, assume that v_{in} is equal to 1 (this can be achieved after an appropriate scaling, if necessary). Let $\{j_1, j_2, ..., j_k\}$ denote a subset of $\{1, 2, ..., n\}$ representing the circuit output ports of interest. The main goal of this part is to address the following problem.

Problem 1: Whether it is possible to turn on a subset of the switches $\{1, 2, ..., n\}$ so that the following constraints are all satisfied:

$$\begin{aligned} &\left|\text{Re}\{v_{j_p} - v_{j_p}^d\}\right| \leq \varepsilon_{j_p}, \quad p = 1, 2, ..., k \\ &\left|\text{Im}\{v_{j_p} - v_{j_p}^d\}\right| \leq \bar{\varepsilon}_{j_p}, \quad p = 1, 2, ..., k \\ &\left|\text{Re}\{y_{\text{in}} - y_{\text{in}}^d\}\right| \leq \varepsilon, \\ &\left|\text{Im}\{y_{\text{in}} - y_{\text{in}}^d\}\right| \leq \bar{\varepsilon} \end{aligned} \tag{3}$$

where:

- $v_{j_1}^d, ..., v_{j_k}^d$ are given desired voltages at the output ports $j_1, j_2, ..., j_k$, respectively.
- y_{in}^d is the desired input admittance.
- Nonnegative numbers $\varepsilon_{j_p}, \bar{\varepsilon}_{j_p}$, $\forall p = 1, 2, ..., k$, and $\varepsilon, \bar{\varepsilon}$ are given permissible tolerances.

Note that the operators $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ return the real and imaginary parts of complex numbers. Define $\{e_1, e_2, ..., e_n\}$ and $\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{n+1}\}$ to be the sets of standard basis vectors of $\Re^n$ and $\Re^{n+1}$, respectively. Denote also the set of complex number with $\mathbf{C}$. Throughout this paper, the notations $\succ$ and $\succeq$ will be used to show inequalities in the positive definite and positive semi-definite senses, respectively. The following theorem recasts Problem 1 as an optimization problem.

Theorem 1: Minimize the rank of the matrix:

$$\begin{bmatrix} X & \begin{bmatrix} \tilde{\mathbf{v}}^* \\ 1 \end{bmatrix} \\ \begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} & 1 \end{bmatrix} \tag{4}$$

for the variables $X \in \mathbf{C}^{(n+1)\times(n+1)}$ and $\tilde{\mathbf{v}} \in \mathbf{C}^{1\times n}$ subject to the constraints:

$$\left|\text{Re}\{\tilde{\mathbf{v}} e_{j_p} - v_{j_p}^d\}\right| \leq \varepsilon_{j_p}, \quad p = 1, 2, ..., k \tag{5a}$$

$$\left|\text{Im}\{\tilde{\mathbf{v}} e_{j_p} - v_{j_p}^d\}\right| \leq \bar{\varepsilon}_{j_p}, \quad p = 1, 2, ..., k \tag{5b}$$

$$\left|\text{Re}\{\begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} Y_s \tilde{e}_{n+1} - y_{\text{in}}^d\}\right| \leq \varepsilon \tag{5c}$$

$$\left|\text{Im}\{\begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} Y_s \tilde{e}_{n+1} - y_{\text{in}}^d\}\right| \leq \bar{\varepsilon} \tag{5d}$$

$$X_p Y_s \tilde{e}_p = 0, \quad p = 1, 2, ..., n \tag{5e}$$

$$X = X^* \tag{5f}$$

where X_p is the p-th row of the matrix X, for all $p \in \{1, 2, ..., n+1\}$. If the value of the minimum rank is greater than 1, then Problem 1 is not feasible; otherwise, it has a solution which can be extracted from $\tilde{\mathbf{v}}$ by searching for zero entries in this vector and turning on the corresponding switches in the system accordingly.

Proof of necessity: Assume that Problem 1 has a feasible solution, with the output voltage vector $\tilde{\mathbf{v}}$. Given $p \in \{1, 2, ..., n\}$, since the ideal switch p has either a zero current or a zero voltage, the product $v_p^* i_p$ is equal to zero. Therefore, it follows from (2a) that:

$$v_p^* \begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} Y_s \tilde{e}_p = 0 \tag{6}$$

Define X to be:

$$X := \begin{bmatrix} \tilde{\mathbf{v}}^* \\ 1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} \tag{7}$$

Observe that the matrix X defined above, together with the vector $\tilde{\mathbf{v}}$, satisfies all constraints given in (5). More precisely:

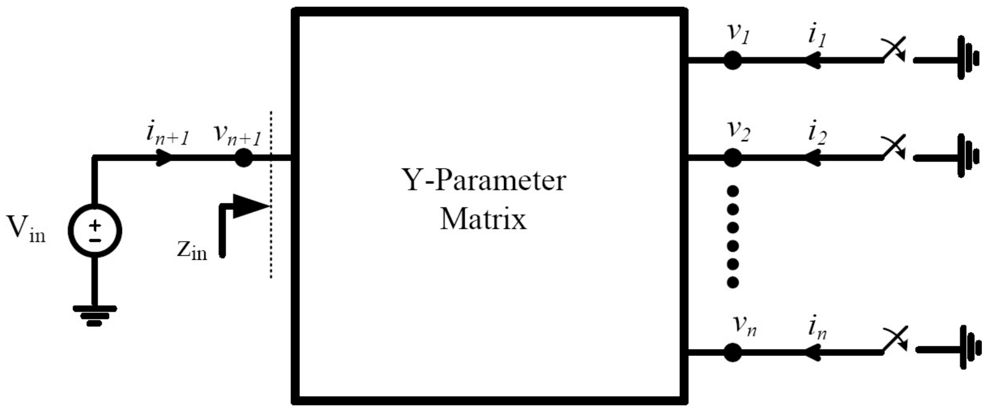

Fig. 1. Circuit model with ideal switches.

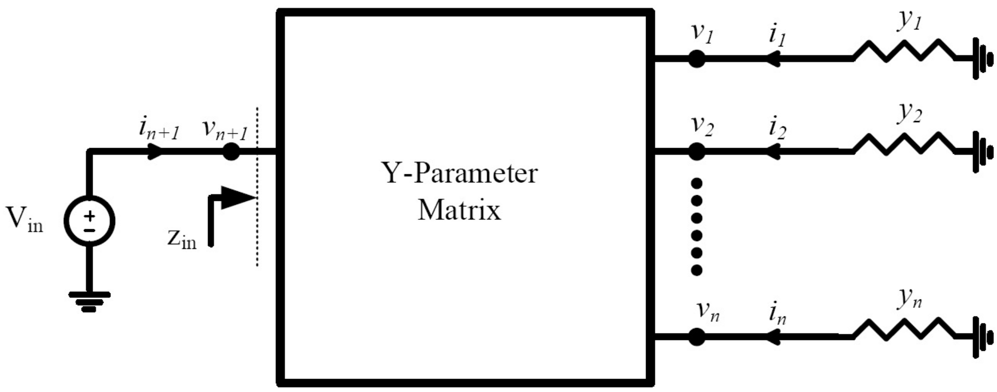

Fig. 2. Circuit model with varactors.

- The conditions (5a), (5b), (5c), (5d) correspond to those given in (3) (by virtue of (2)).
- The relation (5e) holds in light of (6) and (7).
- The condition (5f) follows immediately from the symmetry of the matrix X.
- The rank of the matrix provided in (4) is equal to 1 due to the fact that this matrix can be expressed as the product of two vectors.

Proof of sufficiency: Assume that there exist two matrices X and $\tilde{\mathbf{v}}$ for which the conditions pointed out in the statement of the theorem are all fulfilled. Due to the symmetry of the matrix X, it is straightforward to show that there exists a vector $u \in \mathbf{C}^{n+2}$ such that the matrix given in (4) is equal to uu^* or $-uu^*$ (note that the rank of this matrix is 1, by assumption). On the other hand, the last diagonal entry of the matrix (4), which is equal to 1, does not allow this matrix to be negative semi-definite. Hence:

$$\begin{bmatrix} X & \begin{bmatrix} \tilde{\mathbf{v}}^* \\ 1 \end{bmatrix} \\ \begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} & 1 \end{bmatrix} = uu^* \tag{8}$$

This relation can be simplified to conclude that:

$$\begin{bmatrix} \tilde{\mathbf{v}} & 1 & 1 \end{bmatrix} = \pm u^* \tag{9}$$

As a result, X satisfies the equality:

$$X = \begin{bmatrix} \tilde{\mathbf{v}}^* \\ 1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} \tag{10}$$

The rest of the proof relies on this equation and can be carried out in line with the arguments made in the proof of necessity. The details are omitted here for brevity. ■

Theorem 1 states that checking the feasibility of Problem 1 amounts to solving an optimization problem whose constraints are all linear. However, the rank of a Hermitian

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matrix is to be minimized, which makes the problem nonconvex. The possibility of using a heuristic method to solve this problem will be later discussed in Subsection D. Nonetheless, due to the known NP-harness of the rank minimization problem [17], since there is no guarantee that a heuristic method works satisfactorily, it is desired instead to slightly modify the underlying problem so that it becomes convex. This is the crux of the next subsection.

B. Convex circuit optimization

Consider again the circuit depicted in Figure 1. The nonconvexity of Problem 1 originates from the right side of the circuit that is a collection of switches. Let this part of the circuit be replaced by a reciprocal passive network, namely an RLC network, which will be determined later. This gives rise to Figure 3, where the switches have been supplanted by a passive n-port network, which is to be found in such a way that the design specifications listed in (3) will be satisfied. Roughly speaking, this passive network may be regarded as a collection of some non-ideal switches which are connected to each other in a sophisticated way. Let the objective be formalized now.

Problem 2: Whether there exists a reciprocal passive network (as shown in Figure 3) with an admittance Y at the given frequency ω_0 which allows the following design specifications to be met:

$$\begin{aligned} &\left|\text{Re}\{v_{j_p} - v_{j_p}^d\}\right| \leq \varepsilon_{j_p}, \quad p = 1,2,...,k \\ &\left|\text{Im}\{v_{j_p} - v_{j_p}^d\}\right| \leq \bar{\varepsilon}_{j_p}, \quad p = 1,2,...,k \\ &\left|\text{Re}\{y_{\text{in}} - y_{\text{in}}^d\}\right| \leq \varepsilon, \\ &\left|\text{Im}\{y_{\text{in}} - y_{\text{in}}^d\}\right| \leq \bar{\varepsilon} \end{aligned} \tag{11}$$

Note that the reciprocity condition in the above problem can be translated as the real and imaginary parts of Y are both symmetric. It is desired to show that Problem 2 can be turned into a convex optimization problem of a simple form. In what follows, a lemma is presented which will be used later to prove this interesting result.

Lemma 1: Given symmetric matrices $M, N \in \Re^{n\times n}$, if M is nonsingular, then the following statements are equivalent:

i) M is a positive definite matrix.
ii) $M + NM^{-1}N$ is a positive definite matrix.

Proof: It is evident that (i) implies (ii). Therefore, it only remains to prove the converse statement. To this end, assume that $M + NM^{-1}N$ is a positive definite matrix. Define the following matrices:

$$\begin{aligned} P &:= \begin{bmatrix} M & N \\ N & -M \end{bmatrix}, \\ T &:= \begin{bmatrix} I & -NM^{-1} \\ 0 & I \end{bmatrix}, \\ Q &:= \begin{bmatrix} M + NM^{-1}N & 0 \\ 0 & -M \end{bmatrix} \end{aligned} \tag{12}$$

It is easy to verify that $P = TQT^*$. Denote the number of positive, negative and zero eigenvalues of the symmetric matrix P with η_1, η_2, η_3, respectively. In the same way, denote the same quantities of the matrix Q with the triple $(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)$. Since the matrix T is nonsingular, it follows from Sylvester's Law of Inertia that:

$$(\eta_1, \eta_2, \eta_3) = (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \tag{13}$$

On the other hand, the Hamiltonian structure of the matrix P concludes that:

$$\eta_1 = \eta_2 \tag{14}$$

Furthermore, since the matrix $M + NM^{-1}N$ is positive definite, $\bar{\eta}_1$ is at least equal to n, i.e., the size of this matrix. In light of the equalities (13) and (14), this is possible only when $\eta_1 = \eta_2 = \bar{\eta}_1 = \bar{\eta}_2 = n$. Thus, the matrix Q has n negative eigenvalues. Nonetheless, the negative eigenvalues of this matrix are the same as those of the matrix $-M$; hence, $-M \in \Re^{n\times n}$ has the maximum number of negative eigenvalues. This simply proves that the eigenvalues of M are all positive, which completes the proof. ■

Decompose the matrix Y_s as:

$$Y_s = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{15}$$

where $W_{11} \in \mathbf{C}^{n\times n}$. For given symmetric square matrices A and B of the same dimension with $det(A) \neq 0$, it can be verified that:

$$\begin{aligned} (A + Bi)^{-1} = &(A + BA^{-1}B)^{-1} \\ &+ (A + BA^{-1}B)^{-1}BA^{-1}i \end{aligned} \tag{16}$$

where i stands for the imaginary unit. This identity will be exploited in the next theorem.

Theorem 2: Problem 2 is feasible if and only if there exist two symmetric matrices $M, N \in \Re^{n\times n}$ such that:

$$\begin{bmatrix} \text{Re}\{W_{11}\} - M & N \\ N & M \end{bmatrix} \succ 0 \tag{17}$$

and:

$$\begin{aligned} &\left|\text{Re}\{W_{21}(M + Ni)e_{j_p} + v_{j_p}^d\}\right| \leq \varepsilon_{j_p}, \quad p = 1,2,...,k \\ &\left|\text{Im}\{W_{21}(M + Ni)e_{j_p} + v_{j_p}^d\}\right| \leq \bar{\varepsilon}_{j_p}, \quad p = 1,2,...,k \\ &\left|\text{Re}\{W_{21}(M + Ni)W_{12} - W_{22} + y_{\text{in}}^d\}\right| \leq \varepsilon \\ &\left|\text{Im}\{W_{21}(M + Ni)W_{12} - W_{22} + y_{\text{in}}^d\}\right| \leq \bar{\varepsilon} \end{aligned} \tag{18}$$

Moreover, if there exist such matrices M, N satisfying the above constraints, then one candidate for the admittance matrix Y is:

$$Y = (M + Ni)^{-1} - W_{11} \tag{19}$$

Proof: To tackle Problem 2, one can write the following sets of equations:

$$\begin{aligned} \begin{bmatrix} \tilde{\mathbf{i}} & i_{n+1} \end{bmatrix} &= \begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} Y_s \\ \begin{bmatrix} \tilde{\mathbf{i}} & i_{n+1} \end{bmatrix} &= -\begin{bmatrix} \tilde{\mathbf{v}} & 1 \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & -y_{\text{in}} \end{bmatrix} \end{aligned} \tag{20}$$

These equations can be combined to conclude that:

$$\begin{aligned} &\tilde{\mathbf{v}}(W_{11} + Y) + W_{21} = 0, \\ &\tilde{\mathbf{v}}W_{12} + W_{22} - y_{\text{in}} = 0 \end{aligned} \tag{21}$$

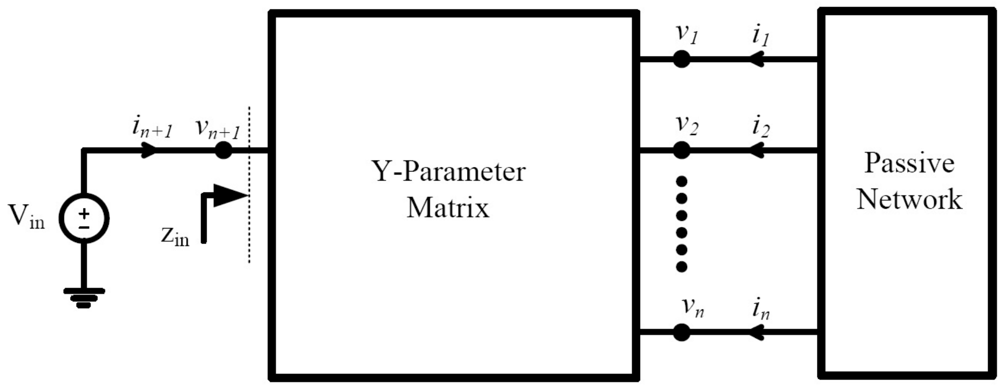

Fig. 3. A modified version of the circuit given in Figure 1 in which a passive network takes the place of the switches.

Thus:

$$\begin{aligned} \tilde{\mathbf{v}} &= -W_{21}\tilde{Y}, \\ y_{\text{in}} &= -W_{21}\tilde{Y}W_{12} + W_{22} \end{aligned} \tag{22}$$

where:

$$\tilde{Y} := (W_{11} + Y)^{-1} \tag{23}$$

(the invertibility of $W_{11} + Y$ follows from the passivity of Y and the strict passivity of Y_s). Now, write $\tilde{Y}$ as $M + Ni$, for some proper real-valued symmetric matrices M and N. It is straightforward to verify that the equations (22) and $\tilde{Y} = M + Ni$ lead to the equivalence of the constraints given in (18) and those provided in Problem 2. Hence, it remains to show that the passivity of the network with the admittance Y at the frequency ω_0 is tantamount to the condition (17). To this end, notice that the passivity constraint on Y can be interpreted as the condition $\text{Re}\{Y\} \succeq 0$ [1]. On applying the identity (16) to the equation (23), this passivity constraint can be expressed as:

$$\begin{aligned} \text{Re}\{Y\} &= \text{Re}\{\tilde{Y}^{-1} - W_{11}\} \\ &= (M + NM^{-1}N)^{-1} - \text{Re}\{W_{11}\} \succeq 0 \end{aligned} \tag{24}$$

On the other hand, since W_{11} is a principal submatrix of the strictly passive matrix Y_s, the quantity $\text{Re}\{W_{11}\}$ is positive definite. Thus, the above passivity constraint can be rearranged as:

$$(M + NM^{-1}N)^{-1} \succeq \text{Re}\{W_{11}\} \succ 0 \tag{25}$$

Two properties can be extracted from this relation:

- First, Lemma 1 yields that if the above condition is satisfied, then:
$$M \succ 0 \tag{26}$$
- Second, the constraint (25) can be manipulated to arrive at:
$$Re\{W_{11}\}^{-1} \succeq M + NM^{-1}N \tag{27}$$
or equivalently:
$$\big(Re\{W_{11}\}^{-1} - M\big) - NM^{-1}N \succeq 0 \tag{28}$$

The Schur's complement formula asserts that the conditions (26) and (28) are identical to the constraint (17). This completes the proof. ■

Theorem 2 states that Problem 2 is equivalent to a linear matrix inequality (LMI) feasibility problem, which can be handled efficiently using a proper software tool such as YALMIP or SOSTOOLS [19], [20]. This signifies that replacing switches with a passive network facilitates the circuit design, at the cost of complicating its implementation in practice. In the case when it is strictly required to design a collection of switches, Theorem 2 is still useful. Indeed, infeasibility of Problem 2 implies infeasibility of Problem 1. As a result, one can regard the LMI problem proposed in Theorem 2 as a sanity test for checking the feasibility of Problem 1.

Assume that Problem 2 is feasible and, therefore, an admittance matrix Y (at the frequency ω_0) is obtained by means of solving the feasibility problem given in Theorem 2. The next step is to design a reciprocal passive (RLC) network whose corresponding admittance transfer function at the frequency ω_0 is equal to Y. This can be accomplished systematically using the existing methods in the literature [1], [2].

C. Continuous decoupled circuit optimization

The main issue with the admittance matrix Y obtained in Theorem 2 is that its corresponding RLC network could potentially have so many components, which impede its implementation. To circumvent this drawback, one can impose a sparsity constraint on Y, saying that it must be (nearly) diagonal. A diagonal Y converts the circuit model to the one depicted in Figure 2, where y_j in the figure denotes the (j, j) entry of Y for all $j \in \{1, 2, ..., n\}$. It is noteworthy that supplanting an ideal switch with a varactor significantly increases the likelihood that the design specifications (3) be feasible. Define Problem 3 to be the same as Problem 2, but under the additional constraint of the diagonality of Y. It will be shown in the sequel that Problem 3 is non-convex;

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however, there is a good heuristic method for this problem, as tested on several practical examples.

Theorem 3: Minimize the rank of the matrix:

$$\begin{bmatrix} \bar{P} & I \\ I & P \end{bmatrix} \tag{29}$$

for symmetric matrices $M, N \in \Re^{n\times n}$ and diagonal matrices $D_1, D_2 \in \Re^{n\times n}$ subject to the constraints given in (18) and:

$$\begin{aligned} D_1 &\geq 0 \\ M &\succ 0 \end{aligned} \tag{30}$$

where P is provided in (12) and:

$$\bar{P} := \begin{bmatrix} D_1 + \text{Re}\{W_{11}\} & -D_2 - \text{Im}\{W_{11}\} \\ -D_2 - \text{Im}\{W_{11}\} & -D_1 - \text{Re}\{W_{11}\} \end{bmatrix} \tag{31}$$

If the value of the minimum rank is greater than $2n$, then Problem 3 is infeasible; otherwise, it is feasible and a candidate for the diagonal admittance matrix Y can be recovered as follows:

$$Y = D_1 + D_2 i \tag{32}$$

Proof: When there is no diagonality constraint on the matrix Y, a necessary and sufficient condition for the existence of a desirable network is provided in Theorem 2. Hence, it suffices to somehow include this extra constraint. For this purpose, write Y as D_1+D_2i, where D_1 and D_2 are required to be diagonal. It results from the equation (19) that:

$$D_1 + D_2 i + W_{11} = (M + Ni)^{-1} \tag{33}$$

Applying the identity (16) to the above equation yields that:

$$\begin{aligned} D_1 + \text{Re}\{W_{11}\} &= (M + NM^{-1}N)^{-1} \\ D_2 + \text{Im}\{W_{11}\} &= (M + NM^{-1}N)^{-1}NM^{-1} \end{aligned} \tag{34}$$

These equations can be written in the matrix form as:

$$\begin{bmatrix} D_1 + \text{Re}\{W_{11}\} & -D_2 - \text{Im}\{W_{11}\} \\ -D_2 - \text{Im}\{W_{11}\} & -D_1 - \text{Re}\{W_{11}\} \end{bmatrix} = \begin{bmatrix} M & N \\ N & -M \end{bmatrix}^{-1} \tag{35}$$

or equivalently $\bar{P} = P^{-1}$. On the other hand:

$$\begin{bmatrix} \bar{P} & I \\ I & P \end{bmatrix} = \begin{bmatrix} I & P^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{P} - P^{-1} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & 0 \\ P^{-1} & I \end{bmatrix} \tag{36}$$

In light of the equality $\bar{P} - P^{-1} = 0$ and the non-singularity of P, the above equation concludes that the rank of the matrix given in (29) is exactly equal to $2n$. So far, it is shown that the diagonality of the matrix Y implies the aforementioned rank constraint. To prove the converse statement, notice that the condition $M \succ 0$ makes the Hamiltonian matrix P nonsingular (see the proof of Lemma 1). This, together with the identity (36), signifies that if the rank of the matrix in (29) is $2n$, then the matrix $P - \bar{P}^{-1}$ must be zero. This result leads to the equation (33), which is the diagonality constraint of the matrix Y. Moreover, one can easily replace the passivity constraint (17) given in Theorem 2 with the condition $D_1 \geq 0$, because the real part of the matrix Y is equal to D_1. This completes the proof. ■

D. Heuristic method for rank minimization

A rank minimization problem is known to be NP hard in general. However, several heuristic methods have been proposed in the literature to relax the problem to a convex one, whose solution may be identical or near to that of the original problem [10], [18]. In particular, the works [10], [11] suggest minimizing the nuclear norm, i.e., the sum of the singular values, of the matrix whose rank is to be minimized. This heuristic method works correctly with overwhelming probability for a broad class of random rank-minimization problems. Although the rank minimization problems given in Theorems 1 and 3 may not lie into that category of well-behaved problems, an extensive simulation was done by the authors to test the efficiency of this method. It was observed that the heuristic method works perfectly for the optimization problem in Theorem 3 if there is only constraints on y_{in}. When there are constraints on the output voltages, some of the constraints may be violated a trifle. This might be partially due to the fact that the circuit problem is not numerically robust with respect to these voltages, and a small change in the parameters of the network makes the voltages alter noticeably. Note that the nuclear-norm heuristic method may fail to obtain a satisfactory result when applied to Theorem 1, as simulation suggests. Proposing a more effective heuristic method for the optimization problem in the switching case or deriving an alternative optimization problem is left for future search.

E. Generalizations

The circuit problem investigated in this paper may need to be under additional constraints as follows:

- The design specifications given in (3) are only at the frequency ω_0. There could be multiple frequencies, each one associated with similar design requirements.
- There may be extra conditions on the matrix Y in Problems 2 and 3, such as being lossless.
- Some of the output voltages may be required to be sufficiently weak. This introduces norm inequality constraints.

The optimization results obtained in the paper can be straightforwardly modified to encompass the above constraints, together with many others.

IV. Simulation Results

Consider the antenna configuration depicted in Figure 4, which consists of a transmitting dipole antenna (blue bar), a 3x3 array of metal plates (antenna parasitic elements), and a receiving dipole antenna located at the far field (green bar). There are 14 ports in this figure as follows:

- Port 1 corresponds to the transmitting antenna.
- Ports 2 to 13 are intended to change the boundary condition of the transmitting antenna.
- Port 14 acts as a receiving antenna sampling the radiation pattern of the transmitting antenna at a specific angle in the far field

The objective is to find optimum impedance values for the parasitic elements such that the received power, the antenna

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gain, or the antenna input impedance satisfy a specific set of constraints. For this purpose, the circuit model of the antenna system is extracted at the desired frequency 3.5GHz (using localized differential lumped ports) by means of the electromagnetic software IE3D [21]. This model can be any of the circuits given in Figures 1, 2, 3, depending on how the impedances of the parasitic elements are designed. Note that n is equal to 13 in this example, and that $v_{n+1} = v_{14} = 1$.

Three important goals in a typical antenna problem are (i) received power maximization, (ii) antenna gain maximization, (iii) input impedance matching. Tackling these problems is central to this section, which is carried out in the sequel.

Notice that the power at the receiving antenna is proportional to the 2-norm of v_1 raised to the second power. Since maximizing the 2-norm of any quantity is normally a non-convex problem, it is desired to maximize the 1-norm of v_1, i.e. $|\text{Re}\{v_1\}| + |\text{Im}\{v_1\}|$. This suggestion is motivated by the close affinity between these two norms. Observe that the direct maximization of $|\text{Re}\{v_1\}|+|\text{Im}\{v_1\}|$ is again a non-convex optimization problem. Nevertheless, one can alternatively perform four (convex) optimizations maximizing the quantities $\text{Re}\{v_1\} + \text{Im}\{v_1\}$, $\text{Re}\{v_1\} - \text{Im}\{v_1\}$, $-\text{Re}\{v_1\} + \text{Im}\{v_1\}$, and $-\text{Re}\{v_1\} - \text{Im}\{v_1\}$, and then determine the maximum solution corresponding to the desirable objective function $|\text{Re}\{v_1\}|+|\text{Im}\{v_1\}|$. Problem 2 is adapted to solve these optimization problems, under the additional constraint that the passive matrix Y to be found has only zero entries in its first row and first column. This requirement reflects the fact that port 1 is aimed to merely measure the antenna gain at the far field and hence it is not practical to change the impedance of this port. The outcome of these convex optimization problems is summarized in Table I, which demonstrates that the optimal value of $|\text{Re}\{v_1\}| + |\text{Im}\{v_1\}|$ is equal to 0.2833 that corresponds to the received power 2.8461×10^{-4} and the antenna gain 13.4480dB. It is interesting to note that this result is obtained by solving four convex optimization problems, each of which is handled by the software CVX [22] in a fraction of second (the simulation was run on a computer with a Pentium IV 3.0 GHz and 3.62 GB of memory).

Now, assume that the objective is to maximize the antenna gain subject to the constraint that the antenna input impedance is equal to the standard value 50Ω. As before, the antenna gain is proportional to the 2-norm of the output voltage v_1 raised to the second power. The non-convexity of the underlying problem suggests maximizing the closely related term $|\text{Re}\{v_1\}| + |\text{Im}\{v_1\}|$. Similar to the previous case, four convex optimization problems are solved, and the results are summarized accordingly in Table II. One can observe that the (near) optimal value of the antenna gain is equal to 13.0630dB.

As the last scenario, the goal is to find a diagonal matrix Y (with its (1,1) entry equal to 0) such that the antenna input impedance is matched with the value 50Ω. The heuristic method given in [11] was applied to Problem 3 to find proper values for the diagonal matrices D_1 and D_2 (recall that $Y = D_1 + D_2 i$). Fortunately, an appropriate solution was found as follows:

$$\begin{aligned} D_1 = \text{diag}[&0, 0.0026, 0.0026, 0.0070, 0.0070, 0.0026, 0.0026, \\ &0.0138, 0.0134, 0.3252, 0.4268, 0.0136, 0.0123] \\ D_2 = \text{diag}[&0, -0.0106, -0.0105, -0.0064, -0.0064, -0.0106, \\ &-0.0105, 0.0215, 0.0217, -0.0050, -0.0036, \\ &0.0227, 0.0205] \end{aligned} \tag{37}$$

which corresponds to the antenna gain 7.7786dB.

V. Conclusions

This paper studies a class of linear networks that appear in circuits, electromagnetics, optics, etc. Given such a linear system, the objective is to find certain parameters of the circuit (system) subject to some design specifications. This circuit problem is tantamount to a rank minimization problem of a simple form. In light of the particular structure of this optimization, it is verified that a celebrated heuristic method works satisfactorily for this non-convex optimization. Moreover, it is proved that a slight modification of the underlying problem makes the corresponding optimization problem convex. This modification is pragmatic and solely complicates the device implementation. The results of the current work are derived using available techniques in the control theory.

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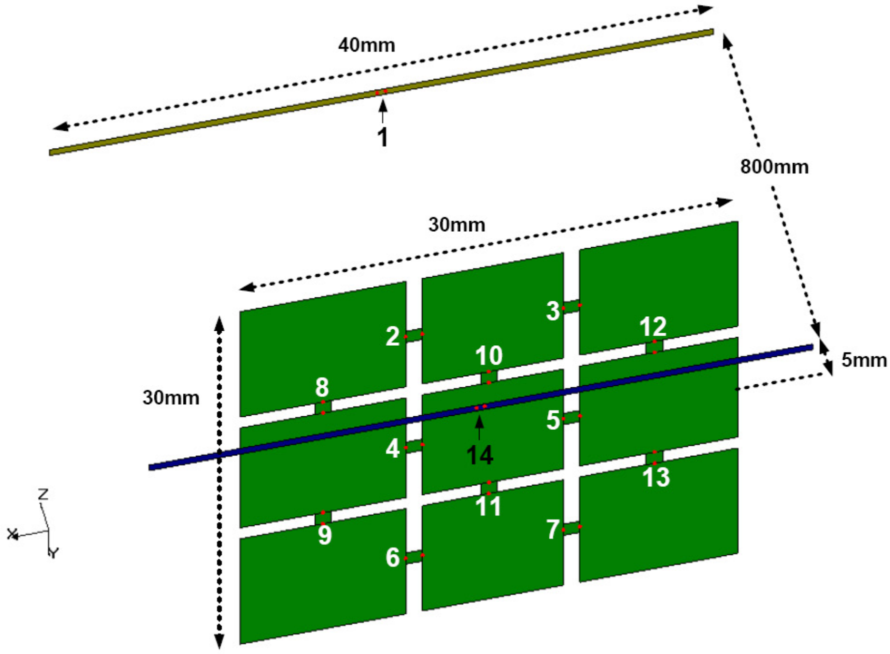

Fig. 4. Antenna problem studied in Section IV.

TABLE I

RECEIVED POWER MAXIMIZATION USING A PASSIVE NETWORK

Objective function	Optimal value	Voltage at port 1	Transmitted power	Received power	Antenna gain	CPU time
Re$\{v_1\}$+Im$\{v_1\}$	0.0491	$-0.0126 - 0.0365i$	0.0199	1.0500×10^{-5}	$4.4804\ (6.5132dB)$	0.34sec
Re$\{v_1\}$−Im$\{v_1\}$	0.1423	$-0.0126 + 0.1297i$	0.0553	1.1958×10^{-4}	$18.3617\ (12.6391dB)$	0.55sec
−Re$\{v_1\}$+Im$\{v_1\}$	0.1902	$0.1536 - 0.0365i$	0.0737	1.7552×10^{-4}	$20.2232\ (13.0585dB)$	0.56sec
−Re$\{v_1\}$−Im$\{v_1\}$	0.2833	$0.1536 + 0.1297i$	0.1093	2.8461×10^{-4}	$22.1208\ (13.4480dB)$	0.63sec

TABLE II

ANTENNA GAIN MAXIMIZATION USING A PASSIVE NETWORK

Objective function	Optimal value	Voltage at port 1	Transmitted power	Received power	Antenna gain	CPU time
Re$\{v_1\}$+Im$\{v_1\}$	0.0432	$-0.0192 - 0.0240i$	0.0100	6.6538×10^{-6}	$5.6500\ (7.5205dB)$	0.78sec
Re$\{v_1\}$−Im$\{v_1\}$	0.0579	$-0.0192 + 0.0387i$	0.0100	1.3158×10^{-5}	$11.1733\ (10.4818dB)$	0.87sec
−Re$\{v_1\}$+Im$\{v_1\}$	0.0674	$0.0434 - 0.0240i$	0.0100	1.7337×10^{-5}	$14.7211\ (11.6794dB)$	0.86sec
−Re$\{v_1\}$−Im$\{v_1\}$	0.0821	$0.0434 + 0.0387i$	0.0100	2.3841×10^{-5}	$20.2442\ (13.0630dB)$	0.82sec

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