Congestion Control Algorithms from Optimal Control Perspective

Javad Lavaei, John C. Doyle and Steven H. Low

Abstract— **This paper is concerned with understanding the connection between the existing Internet congestion control algorithms and the optimal control theory. The available resource allocation controllers are mainly devised to derive the state of the system to a desired equilibrium point and, therefore, they are oblivious to the transient behavior of the closed-loop system. This work aims to investigate what dynamical functions the existing algorithms maximize (minimize). In particular, it is shown that there exist meaningful cost functionals whose minimization leads to the celebrated primal, dual and primal/dual algorithms. An implication of this result is that a real network problem may be solved by regarding it as an optimal control problem on which some practical constraints, such as a real-time link capacity constraint, are imposed.**

I. Introduction

There has been a growing interest in studying the Internet congestion control ever since the first congestion collapse occurred [1]. Many algorithms have been proposed in the literature to allocate the available network resources in a fair manner among the competing users, without overloading the network. The main idea behind all these algorithms is more or less the same: each user measures some feedback signal, such as packet loss or queueing delay, and accordingly adapts its transmission rate. Among the existing transmission control protocols (TCP) for congestion control, one can name TCP-Tahoe, Reno, New Reno, and Vegas [2], [3]. More complete surveys of this topic can be found in [4], [5] and [6].

The seminal works [7] and [8] sparked remarkable process in mathematical modeling and analysis of the Internet congestion control. This advancement is due to the convex programming theory, which allows to solve a utility maximization problem by means of the Lagrangian technique. The available resource allocation algorithms, such as primal, dual and primal/dual algorithms, are particularly designed to solve the underlying problem in a distributed way asymptotically. In other words, these algorithms guarantee that the asymptotic transmission rate of each user is the fairest rate which can be utilized without congesting the network. Having regarded the network as a system, this result implies that the control system possesses a unique globally asymptotically stable equilibrium point that corresponds to the solution of the static utility maximization problem. Nonetheless, it is not clear how well the system operates during its transient time. As a result, the capacity link constraints can, for instance, be violated in this period. Furthermore, these algorithms have not been derived in such a systematic way that they can be generalized routinely to include real-time constraints such as a link capacity requirement. This work aims to revisit the congestion control problem from the standpoint of the optimal control theory.

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This paper proves that the controllers proposed by primal, dual and primal/dual algorithms all maximize some meaningful dynamical behaviors. More precisely, there exist natural cost functionals whose minimization (maximization) leads to these celebrated controllers. This result opens the possibility of tackling network problems directly as optimal control problems, which not only take the dynamics into account, but which also allow to impose physical constraints. Other applications of dealing with cost functionals directly are in deducing the stability of the control system for free, gaining insight into how to perform joint routing and congestion control, etc. It is noteworthy that the development of this work relies on the inverse optimal control theory, which has a very ancient history [9], [10].

The paper is organized as follows. Some preliminaries are provided in Section II, followed by an outline of the motivations of this work in Section III. The dual algorithm is studied in Section IV using optimal control techniques, which is extended to the primal and primal/dual algorithms in Sections V and VI. Finally, some concluding remarks are drawn in Section VII.

II. Preliminaries

Consider a network with the set of sources $\mathcal{S}$ and the set of links $\mathcal{L}$, where each source is identified by an origin and a destination between which data can be transferred. For every $r \in \mathcal{S}$, let x_r denote the transmission rate corresponding to source r and $\mathcal{L}(r)$ denote the collection of links belonging to its fixed route. Assume that each link $l \in \mathcal{L}$ has a finite capacity c_l. Form a vector of transmission rates, denoted by $\mathbf{x}$, where its r-th element is equal to x_r for all $r \in \mathcal{S}$. The resource allocation problem is concerned with solving the following optimization:

$$\max_{\mathbf{x}} \sum_{r \in \mathcal{S}} U_r(x_r) \tag{1}$$

subject to:

$$\begin{aligned} \sum_{r:\ l \in \mathcal{L}(r)} x_r \leq c_l, \quad & \forall\ l \in \mathcal{L} \\ x_r \geq 0, \quad & \forall\ r \in \mathcal{S} \end{aligned} \tag{2}$$

where $U_r : \Re \rightarrow \Re$, $r \in \mathcal{S}$, is a strictly concave, increasing and twice differentiable utility function associated with source r. Define R to be a routing matrix whose (l, r) entry ($r \in \mathcal{S},\ l \in \mathcal{L}$) is equal to 1 if $l \in \mathcal{L}(r)$, and is 0 otherwise. Define also the

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aggregate flow rate y_l, the route price q_r and the Lagrangian $L(\mathbf{x},\mathbf{p})$ as follows:

$$
\begin{aligned}
y_l &:= \sum_{r:\ l\in\mathcal{L}(r)} x_r, \quad l\in\mathcal{L}\\
q_r &:= \sum_{l\in\mathcal{L}(r)} p_l, \quad r\in\mathcal{S}\\
L(\mathbf{x},\mathbf{p}) &:= \sum_{r\in\mathcal{S}} U_r(x_r) - \sum_{l\in\mathcal{L}} p_l\left(y_l - c_l\right)
\end{aligned}
\tag{3}
$$

where $\mathbf{p}$ is the vector of Lagrange multipliers $p_l,\ l\in\mathcal{L}$. The Karush-Kuhn-Tucker (KKT) conditions for the utility maximization problem are:

$$
\begin{aligned}
U'(x_r) &= q_r\\
p_l(y_l - c_l) &= 0\\
y_l - c_l &\leq 0\\
x_r, p_l &\geq 0
\end{aligned}
\tag{4}
$$

for all $l\in\mathcal{L}$ and $r\in\mathcal{S}$. Having assumed that R has full row rank, the above KKT equations have a unique solution $(\mathbf{x}^*,\mathbf{p}^*)$ [5]. Since each user r must obtain its optimal transmission rate x_r^* in terms of its local information, a number of distributed algorithms have been proposed in the literature to enable every user to adaptively find its optimal transmission rate. One of these algorithms is briefly outlined in the sequel.

A. Dual algorithm

Assume that each link $l\in\mathcal{L}$ updates its associated price p_l based on the following rule:

$$
\dot{p}_l(t) = h_l(p_l(t))(y_l(t)-c_l)^+_{p_l(t)} \tag{5}
$$

where $h_l:\Re\to\Re^+$ is a given non-decreasing continuous function and:

$$
(y_l(t)-c_l)^+_{p_l(t)} = \begin{cases} y_l(t)-c_l & p_l(t)>0\\ \max(y_l(t)-c_l,0) & p_l(t)=0\end{cases} \tag{6}
$$

Moreover, suppose that the user of each source $r\in\mathcal{S}$ is provided by the aggregate price along its route to update its transmission rate as below:

$$
x_r(t) = U_r'^{-1}(q_r(t)) \tag{7}
$$

It is well-known that the interconnected system specified by (5) and (7) is globally asymptotically stable with the unique equilibrium point $(\mathbf{x}^*,\mathbf{p}^*)$ [5].

III. Motivation and problem formulation

The main idea behind the existing congestion control algorithms is to contrive a distributed control system which has a unique equilibrium point $(\mathbf{x}^*,\mathbf{p}^*)$ that is globally asymptotically stable. However, this interesting technique is oblivious to the transient behavior of the system and merely targets its steady-state behavior. As a result, the capacity link constraints may be violated during the transient time. Moreover, these indirect congestion control algorithms cannot be generalized systematically. For instance, it is pragmatic to impose a buffer size constraint or to assume that each source has a certain amount of data to transfer. These practical constraints, along with many other ones, cannot be incorporated into the aforementioned algorithms in light of the fact that these algorithms essentially rely on the static utility maximization problem to which these constraints do not apply. Having regarded the network as a system with a specific topology, a question arises as to whether one can define an optimal control problem whose solution leads to a distributed controller which solves the utility maximization problem. This paper aims to show that the answer to this fundamental question is affirmative, and that working directly with the network problems in the context of optimal control allows the designer to incorporate other physical constraints and deduce some properties for free such as stability.

The objective is to prove that the updating policies proposed by the primal, dual and primal/dual algorithms can all be obtained by minimizing appropriate cost functionals which take the transient response of the system into account. Nevertheless, it is well-understood that even though an optimal control problem normally has a unique solution, there might be an infinite number of optimal control problems which all lead to the same solution. For instance, consider the simple first-order system $\dot{p}(t)=x(t)$, where $p(t)$ and $x(t)$ are its state and input, respectively. Note that although $x(t)$ is a standard notation for representing the state of a system, this paper needs to use this notation to denote the input of a system (as it corresponds to the transmission rate that acts as an input). Given a positive scalar k and a positive time instance T, there exists a unique controller that minimizes the cost functional:

$$
\int_0^T\left(\frac{x(t)^2}{k}+kp(t)^2\right)dt + p(T)^2 \tag{8}
$$

This controller turns out to be $x(t)=-kp(t)$. However, there are other cost functionals whose minimization leads to this controller. For example, the trivial term $(x(t)+kp(t))^2$ can be added to the integrand of the above cost functional without altering the optimal solution. It can be shown in this example that all such functionals can be characterized systematically, provided the terminal cost is fixed as $p(T)^2$. To be more precise, assume that the minimization of the cost functional:

$$
\int_0^T g(p(t),x(t))dt + p(T)^2 \tag{9}
$$

yields the controller $x(t)=-kp(t)$, where $g(p(t),x(t))$ is some appropriate function. One can verify that there exist a function $\hat{g}(p(t),x(t))$ and a scalar μ such that:

$$
g(p(t),x(t)) = \mu + \hat{g}(p(t),x(t)) + \frac{x(t)^2}{k} + kp(t)^2 \tag{10}
$$

where $\hat{g}(p(t),x(t))$ is equal to zero along all trajectories of the optimal closed-loop system. This simple toy example implies that there are an infinite number of cost functionals which solve the inverse optimal problem; nonetheless, they all share some key part that determines the trade-off between the state and the input which has caused the optimal controller to be identical to the given one.

The above discussion signifies that there may be numerous cost functionals associated with the static utility maximization problem. The primary objective is to identify their common part which has meaningful physical interpretations. It will be later shown that there is a close parallel (term by term) between the cost functionals solving the utility maximization problem and the ones characterized in (9) and (10).

IV. Optimal control for dual algorithm

Having provided each user r with its route price that is obtained based on some pre-specified rule, assume that the user is required to find the best updating policy to adjust its transmission rate x_r. This hypothesis implies that the following dynamical system exists in the core of the network to generate link prices:

$$\dot{p}_l(t) = h_l(p_l(t))(y_l(t) - c_l)^+_{p_l(t)}, \quad l \in \mathcal{L} \tag{11}$$

where $\mathbf{p}(t)$ and $\mathbf{x}(t)$ are the state and the input of the system, respectively. It is desired to find a cost functional whose minimization leads to the local controllers:

$$x_r(t) = U_r'^{-1}(q_r(t)), \quad r \in \mathcal{S} \tag{12}$$

A. Simple illustrative example

Before handling the problem in the general case, let the main ideas be elucidated in a very simple example. As a trivial but illustrative case, assume that:

- The network has only one source and one link.
- The capacity of the link is equal to 1.
- The utility function $U(x)$ is equal to $-0.5(x-4)^2$ if $x \in [0, 3]$.
- The function $h(p)$ is identical to 1.

Note that since $\mathcal{S}$ and $\mathcal{L}$ each have one element, the indices l and r are omitted. Moreover, although the utility function $U(x)$ is defined only on the interval of interest $[0, 3]$, it can be extended smoothly to the entire interval $[0, \infty)$. For simplicity, suppose that the value of the initial price $p(0)$ is chosen so that the transmission rate $x(t)$ always stays in the interval $[0, 3]$, and that the price $p(t)$ never hits zero. The problem now reduces to finding a cost functional whose minimization leads to the controller:

$$x(t) = -q(t) + 4 \tag{13}$$

for the system:

$$\dot{q}(t) = x(t) - 1 \tag{14}$$

In order to eliminate the constant terms in the above equations, introduce the change of variables:

$$\begin{aligned} \bar{x}(t) &= x(t) - 1 \\ \bar{q}(t) &= q(t) - 3 \end{aligned} \tag{15}$$

In the new coordinates, the system and the controller turn out to be $\dot{\bar{q}}(t) = \bar{x}(t)$ and $\bar{x}(t) = -\bar{q}(t)$, respectively. This control system has been studied in the toy example of the previous section (assuming $k = 1$), for which the following cost functional was obtained:

$$\int_0^T \left(\bar{x}(t)^2 + \bar{q}(t)^2\right) dt + \bar{q}(T)^2 \tag{16}$$

One can rewrite the above expression in terms of the original variables to obtain:

$$\int_0^T \left((x(t)-1)^2 + (q(t)-3)^2\right) dt + (q(T)-3)^2 \tag{17}$$

To relate the terms in the above functional to the static utility maximization problem, notice that:

$$\begin{aligned} 3 - q(t) &= U'^{-1}(q(t)) - 1 = \arg\max_v L(v, q(t)) \\ (q(T)-3)^2 &= 2\max_v L(v, q(T)) + 9 \end{aligned} \tag{18}$$

Substituting the above relations into (17), one can conclude that minimizing the cost functional given below leads to the dual controller:

$$\begin{aligned} &\frac{1}{2}\int_0^T \left((x(t)-c)^2 + \left(\arg\max_v L(v, q(t)) - c\right)^2\right) dt \\ &+ \max_v L(v, q(T)) \end{aligned} \tag{19}$$

As can be inferred from the toy example in Section III, every other cost functional that is able to solve the underling inverse optimal problem includes the integrand of the above functional, in addition to some trivial terms, provided its terminal cost is chosen as above. This result will be generalized in the sequel, and the interpretation of the individual terms appearing in this cost functional will then be discussed in detail.

B. General case

The next theorem extends the above-mentioned results to the general case.

Theorem 1: Given $T > 0$, the decentralized controller given in (12) minimizes the cost functional:

$$\begin{aligned} \min_{\mathbf{x}(t)} \Bigg\{ &\frac{1}{2}\int_0^T \sum_{l \in \mathcal{L}} \left\{Y_l(y_l(t), p_l(t)) + Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t))\right\} dt \\ &+ \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) \Bigg\} \end{aligned} \tag{20}$$

for the system (11), where:

$$Y_l(\alpha, p_l(t)) := (\alpha - c_l) h_l(p_l(t))(\alpha - c_l)^+_{p_l(t)} \tag{21}$$

for every $\alpha \in \Re$, $l \in \mathcal{L}$, and:

$$\tilde{\mathbf{y}}(\mathbf{p}(t)) := R \times \arg\max_{\mathbf{v}(t)} L(\mathbf{v}(t), \mathbf{p}(t)) \tag{22}$$

($\tilde{y}_l(\mathbf{p}(t))$ is equal to the l-th entry of $\tilde{\mathbf{y}}(\mathbf{p}(t))$).

The proof of Theorem 1 is provided in the appendix. The cost functional given in this theorem has several interesting features that will be spelled out next.

Consider the price vector $\mathbf{p}(t)$ at a time instance $t \in [0, T]$. The best transmission rates that the users may utilize at this time can be obtained by maximizing the term $L(\mathbf{v}(t), \mathbf{p}(t))$ over all possible $\mathbf{v}(t)$'s. In other words, $\arg\max_{\mathbf{v}(t)} L(\mathbf{v}(t), \mathbf{p}(t))$ is indeed the optimal instantaneous transmission rates that the system can accept given its current link prices. As a result, the terminal cost $\max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T))$ resembles the static Lagrangian at time T, but it is maximized over all possible transmission rates to evaluate the potential of the system given its final price $\mathbf{p}(T)$. In other words, a variant of the static utility maximization problem is mainly integrated into the final cost (and partially incorporated into the integrand to take care of the transient behavior). On the other hand, the integrand has two terms $Y_l(y_l(t), p_l(t))$ and $Y_l(\tilde{y}_l(\mathbf{p}(t), p_l(t))$, each of which has a physical interpretation. The term $-Y_l(y_l(t), p_l(t))$ can be regarded as the actual l-th link utility at time t, by virtue of the following observations:

- If $p_l(t)$ is nonzero, then $Y_l(y_l(t), p_l(t))$ is proportional to the quadratic term $(y_l(t) - c_l)^2$, which implies that in order not to over-utilize or under-utilize the network, the best strategy is to maintain the flow rate $y_l(t)$ precisely at the capacity of the link.
- If $p_l(t)$ is zero, then $Y_l(y_l(t), p_l(t))$ indicates that the optimal utilization of the link corresponds to employing a flow rate below the link capacity.

Furthermore, $-Y_l(\tilde{y}_l(\mathbf{p}(t), p_l(t))$ can be envisaged as the virtual l-th link utility at time t due to the fact that $\tilde{y}_l(\mathbf{p}(t))$ is the optimal transmission rate over the l-th link given the current price $\mathbf{p}(t)$. To summarize the ideas, the proposed cost functional is natural in the sense it maximizes the sum of the actual and virtual link utilities over the time interval $[0, T)$ and a variant of the static utility function at the final time T.

Corollary 1: For every time instance $T > 0$, the following relation holds:

$$\min_{\mathbf{x}(t)} \left\{ \frac{1}{2} \int_0^T \sum_{l \in \mathcal{L}} \left\{ Y_l(y_l(t), p_l(t)) + Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t)) \right\} dt + \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) \right\} = \max_{\mathbf{v}(0)} L(\mathbf{v}(0), \mathbf{p}(0)) \tag{23}$$

Proof: It follows from the proof of Theorem 1 and the Hamilton-Jacobi-Bellman equation that the expression given in the left side of the equality (23) is identical to the optimal cost-to-go $J(\mathbf{p}(0), 0)$. On the other hand, it is shown in the proof of Theorem 1 that $J(\mathbf{p}(0), 0)$ is equal to the right side of the above equation. This completes the proof. ■

Theorem 1 and corollary 1 assert that there exists a natural cost functional whose minimization leads to the celebrated dual TCP controller, and that the minimum value of this functional is equal to $\max_{\mathbf{v}(0)} L(\mathbf{v}(0), \mathbf{p}(0))$. As pointed out earlier, this term corresponds to the maximum source utility at time $t = 0$ under the given initial price $\mathbf{p}(0)$.

Evidently, there are some cost functionals which trivially solve this problem. For instance, one candidate is as follows:

$$\int_0^\infty \sum_{r \in \mathcal{S}} \left(x_r(t) - U_r'^{-1}(q_r(t)) \right)^2 dt \tag{24}$$

Nevertheless, this cost functional has nothing to do with the original utility maximization problem, and provides no extra information about the system such as its closed-loop stability. In contrast, Theorem 1 proposes a meaningful cost functional, which is a bit involved. A question arises as to whether there exists a simpler cost functional which still conveys meaningful interpretations. To answer this question, notice that the terminal cost given in (20) is a suitable counterpart of the original static utility function. Therefore, it remains to show that the integrand of this functional is essentially required and cannot be simplified. For this purpose, assume that the controller (12) minimizes the cost functional:

$$\min_{\mathbf{x}(t)} \left\{ \int_0^T g(\mathbf{p}(t), \mathbf{x}(t)) dt + \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) \right\} \tag{25}$$

for the system (11), where T is a positive time and $g(\mathbf{p}(t), \mathbf{x}(t))$ is some function. Suppose also that $g(\mathbf{p}, \mathbf{x})$ is continuously differentiable at every point $(\mathbf{p}, \mathbf{x})$ for which $\mathbf{p}$ is strictly positive. Define the optimal cost-to-go function $J(\mathbf{p}, t)$ as follows:

$$J(\mathbf{p}, t) := \int_t^T g(\tilde{\mathbf{p}}(s), \tilde{\mathbf{x}}(s)) ds + \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \tilde{\mathbf{p}}(T)) \tag{26}$$

where:

- $\tilde{\mathbf{p}}(t)$ and $\tilde{\mathbf{x}}(t)$ denote the state and the input of the system (11) under the controller (12).
- The system starts at time t with the initial state $\mathbf{p}$.

Finally, assume that $J(\mathbf{p}, t)$ is continuously differentiable with respect to $\mathbf{p}$ and t.

Theorem 2: Under the assumptions made above, there exist a function $\hat{g}(\mathbf{p}(t), \mathbf{x}(t))$ and a real number μ such that:

$$g(\mathbf{p}(t), \mathbf{x}(t)) = \mu + \hat{g}(\mathbf{p}(t), \mathbf{x}(t)) + \frac{1}{2} \sum_{l \in \mathcal{L}} \left\{ Y_l(y_l(t), p_l(t)) + Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t)) \right\} \tag{27}$$

where the function $\hat{g}(\mathbf{p}(t), \mathbf{x}(t))$ is identically zero along all trajectories of the optimal closed-loop system.

The proof of Theorem 2 is provided in the appendix. Notice that the term $\hat{g}(\mathbf{p}(t), \mathbf{x}(t))$ is a trivial term which provides no information. This quantity can be, for instance, equal to the integrand of the trivial cost functional (24). Ignoring the uninformative terms μ and $\hat{g}(\mathbf{p}(t), \mathbf{x}(t))$, the functional given in Theorem 2 reduces to the one provided in Theorem 1.

It can be observed that the cost functionals characterized in Theorem 2 closely parallel those provided in (9) and (10) for a simple toy example. More specifically:

- $Y_l(y_l(t), p_l(t))$ corresponds to $\frac{x(t)^2}{k}$. This term depends much more weakly on the state, but strongly on the input.
- $Y_l(\tilde{y}_l(t), p_l(t))$ corresponds to $kp(t)^2$, which only penalizes the state.
- The constant term μ exists in both cost functionals.
- $\hat{g}(\mathbf{p}(t), \mathbf{x}(t))$ corresponds to $\hat{g}(p(t), x(t))$, which is an uninformative term and specifies no trade-off between the state and the input.

C. Joint routing and optimal congestion control

Assume that it is desired to accomplish both routing and resource allocation simultaneously. The static version of this problem suggests the following optimization:

$$\max_R \min_{\mathbf{p}} \max_{\mathbf{x}} L(\mathbf{x}, \mathbf{p}) = \max_R L(\mathbf{x}^*, \mathbf{p}^*) \tag{28}$$

In other words, an optimal route is found based on the saddle point of the Lagrangian $L(\mathbf{x}, \mathbf{p})$ (or the equilibrium point $(\mathbf{x}^*, \mathbf{p}^*)$). The same concept has been carried over to the

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dynamical case in which every user is equipped with a local controller. In contrast, it can be inferred from Theorem 1 that the joint routing and resource allocation problem in the dynamical case amounts to the following optimization:

$$\max_{R} \min_{\mathbf{x}(t)} \left\{ \frac{1}{2} \int_0^T \sum_{l\in\mathcal{L}} \{ Y_l(y_l(t), p_l(t)) + Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t)) \} dt + \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) \right\} \tag{29}$$

which is tantamount to (by Corollary 1):

$$\max_{R} \max_{\mathbf{v}(0)} L(\mathbf{v}(0), \mathbf{p}(0)) \tag{30}$$

Comparing (30) with (28), one can observe that Theorem 1 suggests that the optimal route be found in terms of the initial link prices, as opposed to the link prices in the steady-state. This is an interesting and important observation which needs a delicate attention in the dynamic case. To be more precise, this result signifies that as long as the resource allocation problem is required to be solved in a distributed way over the period $[0, T]$, due to the creation of unwanted transient behavior, the best strategy for routing on top of resource allocation may be to search for the best route in terms of the initial link prices, rather than to wait until the system is sufficiently close to its equilibrium point and then find the best route. The reason why a finite horizon control problem is considered here is that users come and go in practice and consequently the topology of the network may change over time. This motivates solving the utility maximization problem for a short period of time, say over the interval $[0, T]$.

D. Stability proof

Another application of the optimal control problem introduced in Theorem 1 is that it automatically concludes the global asymptotic stability of the system (11) under the static controller (12).

Theorem 3: The controller (12) that minimizes the cost functional (20) for the system (11) makes the pair $(\mathbf{x}(t), \mathbf{p}(t))$ converge to the fixed point $(\mathbf{x}^*, \mathbf{p}^*)$.

Proof: The main idea behind the proof is to observe that:

$$\begin{aligned} Y_l(y_l(t), p_l(t)) &\geq 0, \quad \forall t \in [0, T], \ l \in \mathcal{L} \\ Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t)) &\geq 0, \quad \forall t \in [0, T], \ l \in \mathcal{L} \\ \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) &\geq \mathbf{p}^* \end{aligned} \tag{31}$$

and that the state and the input of the closed-loop control system satisfy the equation (by Corollary 1):

$$\frac{1}{2} \int_0^T \sum_{l\in\mathcal{L}} \{ Y_l(y_l(t), p_l(t)) + Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t)) \} dt + \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) = \max_{\mathbf{v}(0)} L(\mathbf{v}(0), \mathbf{p}(0)) \tag{32}$$

By letting T go to infinity, the relations (31) and (32) can be combined to conclude that:

$$\begin{aligned} Y_l(y_l(t), p_l(t)) \to 0 \quad &\text{as} \quad t \to \infty \\ Y_l(\tilde{y}_l(\mathbf{p}(t)), p_l(t)) \to 0 \quad &\text{as} \quad t \to \infty \end{aligned} \tag{33}$$

for every $l \in \mathcal{L}$. As a result:

$$\lim_{t\to\infty} (y_l(t) - c_l)(y_l(t) - c_l)^+_{p_l(t)} = 0, \quad \forall\ l \in \mathcal{L} \tag{34}$$

The proof follows immediately from the above equation. ■

E. Another meaningful cost functional

Roughly speaking, the cost functional proposed in Theorem 1 treats a variant of the static utility function as the terminal cost and defines dynamical utility functions on the links. Another idea would be to define dynamical utility functions on the sources. This idea has been exploited in the next theorem.

Theorem 4: Assume that the weighting functions $h_l(p_l(t))$, $l \in \mathcal{L}$ are all equal to 1. Given $T > 0$, the decentralized controller (12) maximizes the cost functional:

$$\max_{\mathbf{x}(t)} \left\{ \int_0^T \left(\sum_{r\in\mathcal{S}} U_r(x_r(t)) - \max_{\mathbf{v}(t)} L(\mathbf{v}(t), \mathbf{p}(t)) \right) dt - \frac{1}{2} \mathbf{p}(T)^T \mathbf{p}(T) \right\} \tag{35}$$

for the system (11). Furthermore, the maximum of this cost functional is equal to $-\frac{1}{2}\mathbf{p}(0)^T\mathbf{p}(0)$.

Proof: The proof can be carried out in line with that of Theorem 1 after noticing that the optimal cost-to-go function for this control problem is equal to $J(\mathbf{p}, t) = -\frac{1}{2}\mathbf{p}^T\mathbf{p}$. The details are omitted here for brevity. ■

The cost functional proposed in Theorem 4 has an interesting interpretation. The quantity $\max_{\mathbf{v}(t)} L(\mathbf{v}(t), \mathbf{p}(t))$ is equal to the maximum *instantaneous* source utility that the system can provide based on the price $\mathbf{p}(t)$. Hence, the integrand $\sum_{r\in\mathcal{S}} U_r(x_r(t)) - \max_{\mathbf{v}(t)} L(\mathbf{v}(t), \mathbf{p}(t))$ can be regarded as the *relative* source utility function. Having assumed $h_l(p_l(t))$ to be equal to 1, each price $p_l(t)$ can be visualized as the queue size of the l-th router. Thus, the cost functional provided in the theorem aims to maximize the relative utility function over the time interval $[0, T)$ and minimize the sum of routers' queue sizes at the final time T. The maximum of the cost functional, which is equal to the negative half of the sum of the squared queue sizes at $t = 0$, is independent of the route. This property, together with the negative term inside the integrand, does not allow to deduce the stability of the dual control system from this functional for free, or to search for the optimal route.

V. Optimal control for primal algorithm

Let the utility maximization problem stated in Section II be modified as follows:

$$\max_{\mathbf{x}} \left\{ \sum_{r\in\mathcal{S}} U_r(x_r) - \sum_{l\in\mathcal{L}} \int_0^{y_l} f_l(y) dy \right\} \tag{36}$$

where $f_l(y)$ is a barrier function that can be interpreted as the price for transferring data at the rate y on link l. Assume that $f_l(\cdot)$, $l \in \mathcal{L}$, is a non-decreasing, continuous function such that:

$$\int_0^{y_l} f_l(y) dy \to \infty \quad \text{as} \quad y_l \to \infty \tag{37}$$

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Furthermore, assume that $U_r(x_r)$, $r \in \mathcal{S}$, goes to $-\infty$ as x_r approaches zero. Under these assumptions, the above utility maximization problem has a unique solution $\mathbf{x}^*$ at which the gradient of $V(\mathbf{x})$ vanishes, where:

$$V(\mathbf{x}) = \sum_{r\in\mathcal{S}} U_r(x_r) - \sum_{l\in\mathcal{L}} \int_0^{y_l} f_l(y)dy \qquad (38)$$

To obtain the solution $\mathbf{x}^*$ in a distributed way, consider the interconnected system given by:

$$\dot{x}_r(t) = k_r(x_r(t))(U_r'(x_r(t)) - q_r(t)), \quad \forall\, r \in \mathcal{S} \qquad (39)$$

and:

$$p_l(t) = f_l(y_l(t)), \quad \forall\, l \in \mathcal{L} \qquad (40)$$

where $k_r : \Re \rightarrow \Re^+$ is a non-decreasing continuous function. It is known that the point $(\mathbf{x}^*, \mathbf{p}^*)$ is the globally asymptotically stable fixed point of this interconnected system [5]. Thus, the above distributed system can be run to asymptotically solve the static utility maximization problem. The objective is to find the optimal control counterpart of this result. For this purpose, assume that the memoryless system (40) exists in the core of the network to generate the link prices, and that each user deploys a simple integrator to adjust its transmission rate:

$$\dot{x}_r(t) = u_r(t), \quad r \in \mathcal{S} \qquad (41)$$

where $u_r(t)$ is some input signal that needs to be determined. It is noteworthy that $p_l(t)$ is a measured output of this system. The goal is to derive a cost functional for the system (41) whose minimization leads to the decentralized controller:

$$u_r(t) = k_r(x_r(t))\left(U_r'(x_r(t)) - q_r(t)\right), \quad r \in \mathcal{S} \qquad (42)$$

Theorem 5: Given a time instance $T > 0$, the decentralized controller (42) minimizes the cost functional:

$$\min_{\mathbf{u}(t)} \left\{ \frac{1}{2}\int_0^T \Big(\mathbf{u}(t)^T \mathbf{K}(\mathbf{x}(t))^{-1}\mathbf{u}(t) + \nabla V(\mathbf{x}(t))^T \mathbf{K}(\mathbf{x}(t)) \nabla V(\mathbf{x}(t)) \Big) dt - V(\mathbf{x}(T)) \right\} \qquad (43)$$

for the system (41), where:

- $\mathbf{K}(\mathbf{x}(t))$ is a diagonal matrix with the (r,r) diagonal entry $k_r(x_r(t))$ for all $r \in \mathcal{S}$.
- $\mathbf{u}(t)$ is a vector with the r-th entry $u_r(t)$ for all $r \in \mathcal{S}$.
- The symbol ∇ denotes the gradient operator.

Moreover, the minimum of this cost functional is equal to $-V(\mathbf{x}(0))$.

Proof: One can adopt the technique used in Theorem 1 to prove this theorem, after considering the optimal cost-to-go function $J(\mathbf{x}, t)$ as $-V(\mathbf{x})$. ■

As before, the cost functional proposed in the above theorem has some plausible intrinsic properties. For instance, this functional treats the static utility function as a terminal cost, and contains two types of terms accounting for the transient behavior of the system. The term $\nabla V(\mathbf{x}(t))^T \mathbf{K}(\mathbf{x}(t)) \nabla V(\mathbf{x}(t))$ penalizes the nonzero gradient of the objective function $V(\mathbf{x}(t))$ during the transient time (note that the optimal solution of the static utility maximization problem corresponds to the unique point at which the gradient of $V(\mathbf{x})$ vanishes). Besides, the term $\mathbf{u}(t)^T \mathbf{K}(\mathbf{x}(t))^{-1}\mathbf{u}(t)$ or equivalently $\dot{\mathbf{x}}(t)^T \mathbf{K}(\mathbf{x}(t))^{-1}\dot{\mathbf{x}}(t)$ is a measure of users' willingness to alter their transmission rates abruptly. Thus, $\mathbf{K}(\mathbf{x})$ is a weighting function representing the trade-off between the above penalty terms.

In analogy with Theorem 3, the stability of the system (41) under the control (42) is an immediate consequence of Theorem 5. More precisely, since the integrand of the proposed cost functional is always nonnegative and its terminal cost is bounded from below by $-V(\mathbf{x}^*)$, letting T grow towards infinity yields that:

$$\nabla V(\mathbf{x}(t))^T \mathbf{K}(\mathbf{x}(t)) \nabla V(\mathbf{x}(t)) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \qquad (44)$$

or equivalently:

$$||\nabla V(\mathbf{x}(t))|| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \qquad (45)$$

It results from the above relation that the state of the closed-loop system converges to the unique maximizer of the function $V(\mathbf{x})$.

VI. Optimal control for primal/dual algorithm

Consider an interconnected system consisting of the subsystem (11) in the core of the network to generate prices and the subsystem (41) on the user side to adjust the transmission rates. The states of this system are $\mathbf{x}(t)$ and $\mathbf{p}(t)$, while its input (to be found) is $\mathbf{u}(t)$. The goal of this part is to obtain a dynamical cost function whose minimization arrives at the distributed controller (42). The techniques developed earlier can be exploited to tackle this problem; nevertheless, an implausible solution will be attained. More specifically, it can be shown that one such optimal control problem can be defined as follows:

$$\min_{\mathbf{u}(t)} \left\{ \frac{1}{2}\int_0^T \Big(\nabla_x L(\mathbf{x}(t),\mathbf{p}(t))^T \mathbf{K}(\mathbf{x}(t)) \nabla_x L(\mathbf{x}(t),\mathbf{p}(t)) + \mathbf{u}(t)^T \mathbf{K}(\mathbf{x}(t))^{-1}\mathbf{u}(t) - 2\sum_{l\in\mathcal{L}} Y_l(y_l(t), p_l(t)) \Big) dt - L(\mathbf{x}(T),\mathbf{p}(T)) \right\} \qquad (46)$$

where ∇_x denotes the gradient operator with respect to the first argument $\mathbf{x}$. The integrand of this functional is the difference between those given for the primal and dual algorithms (if $\tilde{y}_l(\mathbf{p}(t))$ is identified by $y_l(t)$). However, physical intuition suggests that a good cost functional for this case should be the sum of those obtained for the dual and primal algorithms separately (as opposed to their difference). Indeed, the term $-2\sum_l Y_l(y_l(t), p_l(t))$ in the above cost functional is a measure of link utility (as pointed out earlier) that is minimized, instead of being maximized. This phenomenon can be justified by noticing that the static utility maximization problem is a min-max optimization (as performed on the Lagrangian), whereas the above cost functional is only a min optimization. It is worth noting that the cost functional proposed in Theorem 1 is also a min-max optimization. Hence, it is desired to derive a natural, meaningful cost functional for the primal/dual algorithm in another way.

Lemma 1: The optimization problems:

$$\min_{\mathbf{p}} \max_{\mathbf{x}} L(\mathbf{x}, \mathbf{p}) \tag{47}$$

and:

$$\min_{\mathbf{p},\mathbf{x}} \left\{ -L(\mathbf{x}, \mathbf{p}) + 2 \max_{\mathbf{w}} L(\mathbf{w}, \mathbf{p}) \right\} \tag{48}$$

arrive at the same minimizer, where $\mathbf{w}$ is an auxiliary variable of the same dimension as $\mathbf{x}$.

Proof: The proof is contingent upon the simple observation:

$$-L(\mathbf{x}, \mathbf{p}) + 2 \max_{\mathbf{w}} L(\mathbf{w}, \mathbf{p}) \geq \max_{\mathbf{w}} L(\mathbf{w}, \mathbf{p}) \tag{49}$$

which yields the inequality:

$$\min_{\mathbf{p},\mathbf{x}} \left\{ -L(\mathbf{x}, \mathbf{p}) + 2 \max_{\mathbf{w}} L(\mathbf{w}, \mathbf{p}) \right\} \geq \min_{\mathbf{p}} \max_{\mathbf{x}} L(\mathbf{x}, \mathbf{p}) \tag{50}$$

The proof is completed by noticing that the above inequality turns into an equality if $\mathbf{x} = \mathbf{w} = \mathbf{x}^*$ and $\mathbf{p} = \mathbf{p}^*$. ■

Theorem 6: For any given $T > 0$, the decentralized controller (42) minimizes the cost functional:

$$\begin{aligned} \min_{\mathbf{u}(t)} \Bigg\{ \frac{1}{2} \int_0^T \Big(& \nabla_x L(\mathbf{x}(t), \mathbf{p}(t))^T \mathbf{K}(\mathbf{x}(t)) \nabla_x L(\mathbf{x}(t), \mathbf{p}(t)) \\ & + \mathbf{u}(t)^T \mathbf{K}(\mathbf{x}(t))^{-1} \mathbf{u}(t) + 2 \sum_l Y_l(y_l(t), p_l(t)) \Big) dt \\ & - L(\mathbf{x}(T), \mathbf{p}(T)) + 2 \max_{\mathbf{w}(T)} L(\mathbf{w}(T), \mathbf{p}(T)) \Bigg\} \end{aligned} \tag{51}$$

for the interconnected system given by (11) and (41). In addition, the minimum of this cost functional is equal to $-L(\mathbf{x}(0), \mathbf{p}(0)) + 2 \max_{\mathbf{w}} L(\mathbf{w}, \mathbf{p}(0))$.

Proof: The proof can be carried out in line with that of Theorem 1, after taking the optimal cost-to-go function $J(\mathbf{x}, \mathbf{p}, t)$ as $-L(\mathbf{x}, \mathbf{p}) + 2 \max_{\mathbf{w}} L(\mathbf{w}, \mathbf{p})$. The details are omitted for brevity. ■

As Lemma 1 states, the terminal cost considered in the cost functional (51) is a variant of the static utility function at time T. Furthermore, the integrand of this cost functional is equal to the sum of those given for the primal and dual algorithms (after identifying proper terms). This is an indication of the fact that the primal/dual algorithm integrates the individual features of the primal and dual algorithms. This cost functional allows to derive several properties about stability, routing, etc. as done for the dual algorithm earlier.

VII. Conclusions

This work relates the optimal control theory to the Internet congestion control algorithms. The main motivation for investigating this relationship is that the existing algorithms solve the utility maximization problem at the equilibrium point, and cannot account for the transient behavior of the control system. Therefore, they cannot be modified systematically to incorporate other physical constraints, such as real-time link capacity requirements. In order to substantiate that the optimal control theory provides the right tools to solve a constrained network utility problem in practice, it is shown that there exist natural, meaningful cost functionals whose minimization arrives at the distributed controllers proposed by the primal, dual and primal/dual algorithms. These cost functionals provide useful insights into the optimal closed-loop system; for instance, they automatically conclude the closed-loop stability for free.

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Appendix

Proof of Theorem 1: Define the optimal cost-to-go function $J(\mathbf{p}, t)$, $t \in [0, T]$, as follows:

$$\begin{aligned} J(\mathbf{p}, t) := \min_{\mathbf{x}(s)} \Bigg\{ & \frac{1}{2} \int_t^T \sum_{l \in \mathcal{L}} \big\{ Y_l(y_l(s), p_l(s)) \\ & + Y_l(\tilde{y}_l(\mathbf{p}(s)), p_l(s)) \big\} ds + \max_{\mathbf{v}(T)} L(\mathbf{v}(T), \mathbf{p}(T)) \Bigg\} \end{aligned} \tag{52}$$

where the system starts at time t with an initial state $\mathbf{p}$ whose entries are all nonnegative. The Hamilton-Jacobi-Bellman (HJB) method states that $J(\mathbf{p}, t)$ satisfies the partial differential equation [11]:

$$\begin{aligned} 0 = \frac{\partial J(\mathbf{p}, t)}{\partial t} + \min_{\mathbf{x}} \Bigg\{ & \frac{1}{2} \sum_{l \in \mathcal{L}} \big\{ Y_l(y_l, p_l) \\ & + Y_l(\tilde{y}_l(\mathbf{p}), p_l) \big\} \\ & + \sum_{l \in \mathcal{L}} h_l(p_l)(y_l - c_l)^+_{p_l} \frac{\partial J(\mathbf{p}, t)}{\partial p_l} \Bigg\} \end{aligned} \tag{53}$$

with the boundary condition:

$$J(\mathbf{p}, T) = \max_{\mathbf{v}} L(\mathbf{v}, \mathbf{p}) \tag{54}$$

Solving the HJB differential equation is cumbersome in general. However, it is desired to show that this equation takes the simple solution $J(\mathbf{p}, t) = J(\mathbf{p}, T)$, $\forall t \in [0, T]$ in this problem. To this end, observe that:

$$\tilde{y}_l(\mathbf{p}) = \sum_{r:\ l \in \mathcal{L}(r)} U_r'^{-1}(q_r) \tag{55}$$

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Since $\mathbf{p}$ is a nonnegative vector, the maximum of the Lagrangian $L(\mathbf{v},\mathbf{p})$ (with respect to $\mathbf{v}$) is achieved when:

$$v_r = U_r'^{-1}(q_r), \quad r \in \mathcal{S} \tag{56}$$

where v_r denotes the r-th entry of $\mathbf{v}$, for all $r \in \mathcal{S}$. For the above-mentioned choice of $J(\mathbf{p},t)$, it can be verified that:

$$\begin{aligned} \frac{\partial J(\mathbf{p},t)}{\partial t} &= 0, \\ \frac{\partial J(\mathbf{p},t)}{\partial p_l} &= -\tilde{y}_l(\mathbf{p}) + c_l, \quad \forall\, l \in \mathcal{L} \end{aligned} \tag{57}$$

Using these equalities, one can also check that the input $\mathbf{x}$ given by:

$$x_r = U_r'^{-1}(q_r), \quad r \in \mathcal{S} \tag{58}$$

minimizes the objective functional:

$$\begin{aligned} &\frac{1}{2}\sum_{l\in\mathcal{L}} \left\{ Y_l(y_l,p_l) + Y_l(\tilde{y}_l(\mathbf{p}),p_l) \right\} \\ &+ \sum_{l\in\mathcal{L}} h_l(p_l)(y_l - c_l)^+_{p_l} \frac{\partial J(\mathbf{p},t)}{\partial p_l} \end{aligned} \tag{59}$$

with respect to $\mathbf{x}$. By substituting the equations (57) and (58) into (53), it is straightforward to observe that the equation (53) is satisfied. Hence, the HJB method implies that the controller given in (58) (after replacing (x_r, q_r) with $(x_r(t), q_r(t))$ is an optimal controller for the underlying system. ■

Proof of Theorem 2: In light of the assumptions made right before Theorem 2, one can write the HJB equation for this system as follows:

$$\begin{aligned} 0 = \frac{\partial J(\mathbf{p},t)}{\partial t} + \min_{\mathbf{x}} \Bigg\{ g(\mathbf{p},\mathbf{x}) \\ + \sum_{l\in\mathcal{L}} h_l(p_l)(y_l - c_l)^+_{p_l} \frac{\partial J(\mathbf{p},t)}{\partial p_l} \Bigg\} \end{aligned} \tag{60}$$

where $J(\mathbf{p},t)$ is given in (26). Consider a strictly positive vector $\mathbf{p}$. Taking the derivative of the above expression with respect to x_r, $r \in \mathcal{S}$, yields that:

$$\sum_{l\in\mathcal{L}(r)} h_l(p_l) \frac{\partial J(\mathbf{p},t)}{\partial p_l} = -\frac{\partial g(\mathbf{p},\mathbf{x})}{\partial x_r} \tag{61}$$

Since R has full row rank, the quantities $\frac{\partial J(\mathbf{p},t)}{\partial p_l}$, $l \in \mathcal{L}$, can be uniquely solved in terms of $\frac{\partial g(\mathbf{p},\mathbf{x})}{\partial x_r}$, $r \in \mathcal{S}$. This result, together with the memoryless property of the controller (12), implies that $\frac{\partial J(\mathbf{p},t)}{\partial p_l}$ does not depend on time. Hence, it follows from the HJB equation that $\frac{\partial J(\mathbf{p},t)}{\partial t}$ does not depend on time either. As a result, there exist a scalar μ and a function $f(\mathbf{p})$ such that:

$$J(\mathbf{p},t) = f(\mathbf{p}) - \mu t \tag{62}$$

On the other hand, the boundary condition on the HJB equation states that:

$$J(\mathbf{p},T) = \max_{\mathbf{v}} L(\mathbf{v},\mathbf{p}) \tag{63}$$

Thus, one can conclude that:

$$J(\mathbf{p},t) = \max_{\mathbf{v}} L(\mathbf{v},\mathbf{p}) - \mu(t-T), \quad \forall\; \mathbf{p} > 0 \tag{64}$$

It follows from the continuity of $J(\mathbf{p},t)$ that:

$$J(\mathbf{p},t) = \max_{\mathbf{v}} L(\mathbf{v},\mathbf{p}) - \mu(t-T), \quad \forall\; \mathbf{p} \geq 0 \tag{65}$$

Having written $g(\mathbf{p},\mathbf{x})$ in the form of (27), substituting the above equation into the HJB equation yields that the function $\hat{g}(\mathbf{p}(t),\mathbf{x}(t))$ is equal to zero along all trajectories of the optimal closed-loop system. This completes the proof. ■

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