Back-action limit of linewidth in an optomechanical oscillator

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Laser cooling and amplification of mechanical modes in an optical cavity are now possible with several device geometries. Quantum back action sets the fundamental limit to cooling. It has, however, not been considered in the regenerative oscillator, wherein mechanical amplification overcomes loss, leading to self-sustained, coherent oscillations at rf and microwave rates. In these devices, the spectral purity, as measured by the phase noise, has a back-action contribution that is herein derived and evaluated in recently characterized systems. Beyond its importance as a fundamental stability limit, it can provide a way to observe quantum back action within the context of cavity optomechanics. The analysis is also applied to the case of cooling, and a pump contribution to the cooling limit is derived.

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The motion of a mirror, forming one end of an optical cavity, can be measured with remarkable precision, forming the basis for detection of ultraweak forces [1]. In this process, optical forces have been predicted to modify the motion, setting the ultimate measurement precision [1,2] and even altering the systematic mirror motion through the mechanism of dynamic back action [3]. Unlike static phenomena, such as bistability [4], dynamic back action relies upon cavity delay to induce two distinct regimes of interaction: enhanced mechanical damping and cooling of the mirror motion (red-detuned pump) [5], and parametric instability (blue-detuned pump) [6]. These two regimes have recently been realized experimentally in many laboratories [7]. Cooling [8–12], as well as the related technique of feedback cooling [13,14], is under study as a means to achieve the ground state of a macroscopic mechanical oscillator; the realization of which can enable study of quantum phenomena in the macroscopic realm [15–18]. On the other hand, the parametric instability can be viewed as regenerative oscillation induced by mechanical amplification (negative damping) and has enabled a new optomechanical oscillator [19–21] with oscillation speeds up to microwave rates [22]. Phase noise in these oscillators has been observed to scale inversely with oscillator energy and linearly with temperature [23,24] in accordance with the classical theory of phase noise in a regenerative system.

In addition to phenomena relating to dynamic back action, which is a purely classical mechanism, optomechanical coupling is predicted to induce a quantum back action on the mirror motion. This is well known in the context of gravitational wave detection where the idea of a standard quantum limit of detection was first proposed [1,2]. Likewise, there is a quantum-back-action limit to the cooling process that has recently been theoretically described [25,26]. In this paper, quantum back action in a regenerative, optomechanical oscillator is analyzed. It both sets a fundamental limit of frequency stability in this new class of oscillators, and can provide a way to observe quantum back action. As an aside, the case of cooling is studied briefly and a pump-noise contribution to the back-action cooling limit is noted.

The Hamiltonian $H=\hbar[\omega_o-g(b^\dagger+b)]a^\dagger a+\hbar\Omega_o b^\dagger b$ describes the parametric coupling of the mechanical and optical modes (with coordinates b and a, and eigenfrequencies Ω_o and ω_o). g is the optomechanical coupling parameter [27]. The resulting equations of motion including damping, Langevin operators and pumping are

$$\dot{\beta}=-\frac{\Gamma_o}{2}\beta+\iota g\alpha^\dagger\alpha e^{\iota\Omega_o t}+f(t), \tag{1a}$$

$$\dot{\alpha}=-\frac{\gamma_o}{2}\alpha+\iota g\alpha(b^\dagger+b)+\ell(t)+\iota\sqrt{\gamma_E}Se^{-\iota\Delta\omega t}, \tag{1b}$$

where β and α are slowly varying field operators for the mechanical and optical modes, and are defined through the relations $b\equiv\beta\exp(-\iota\Omega_o t)$ and $a\equiv\alpha\exp(-\iota\omega_o t)$. Damping terms with corresponding Langevin operators $f(t)$ and $\ell(t)$ have also been introduced [28,29] with normalizations provided at the conclusion of this analysis. S is the optical pump amplitude at frequency ω_P, $\Delta\omega\equiv\omega_P-\omega_o$, and γ_E is the external coupling rate. S is normalized so that $|S|^2$ is the rate of photon coupling. With $g=0$ (no optomechanial interaction), $\alpha\equiv\alpha_0+\eta(t)$ where

$$\alpha_0(t)=\frac{\iota\sqrt{\gamma_E}Se^{-\iota\Delta\omega t}}{-\iota\Delta\omega+\gamma_o/2}, \tag{2}$$

$$\eta(t)=\int_{-\infty}^{t}d\tau\, e^{-(\gamma_o/2)(t-\tau)}\ell(\tau). \tag{3}$$

Treating g as a perturbation parameter, time-dependent perturbation gives the following recursion relation for higher-order contributions to α:

$$\alpha_{k+1}=\iota g\int_{-\infty}^{t}d\tau\, e^{-(\gamma_o/2)(t-\tau)}(b^\dagger+b)\alpha_k. \tag{4}$$

In the integrand, the slowly varying amplitudes $[\beta(\tau)$ and $\beta^\dagger(\tau)]$ can be approximated by their values at $\tau=t$ giving the following result:

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$$\alpha_{k+1} = \iota g b^{\dagger} \int_{-\infty}^{t} d\tau\, e^{[-\iota\Omega_o-(\gamma_o/2)](t-\tau)}\alpha_k + \iota g b \int_{-\infty}^{t} d\tau\, e^{[\iota\Omega_o-(\gamma_o/2)](t-\tau)}\alpha_k. \quad (5)$$

This approximation relies upon the optical-cavity damping rate exceeding the characteristic rate of β (i.e., $\Gamma_o \ll \gamma_o$) and is typical of all experimental systems studied to date. The approximation in no way restricts the range of the mechanical eigenfrequency as that portion of the motion is retained within the integral. The results are therefore valid across both the good ($\Omega_o > \gamma_o$) and bad ($\Omega_o < \gamma_o$) cavity limits. For $k=0$, the two terms in Eq. (5) are readily shown to be the Stokes and anti-Stokes waves associated with the "sideband" interpretation of dynamic back action [7,20]. This analysis will include explicitly only contributions from these first-order Stokes and anti-Stokes sidebands and the circulating pump (note that these first-order in g contributions lead to a second-order in g contribution within the mechanical oscillator equation). Higher-order contributions to the pump field are, in fact, important in the regenerative regime as these are responsible for gain saturation. However, these terms are not directly involved in the phase noise analysis. As an aside, Eq. (5) can be used to show that the ratio R of second- to first-order sideband powers is given by $R=\bar{n}_c/n_o$ where $n_o \equiv [(2\Omega_o-\Delta\omega)^2+(\gamma_o/2)^2]/g^2$ and $\bar{n}_c$ is the number of phonons in the coherently oscillating mechanical mode. $\bar{n}_c/n_o<1$ therefore defines the regime of validity of the present analysis. For higher oscillation amplitude, the system can transition into more complex behavior [30]. For sideband-resolved operation in which $\Omega_o \geqslant \gamma_o/2$, the tuning condition $\Delta\omega=\Omega_o$ ensures highest mechanical gain and hence lowest threshold [7]. Under these circumstances, the condition simplifies to $\bar{n}_c < n_o \approx \Omega_o^2/g^2$.

Using $\alpha=\alpha_0+\alpha_1+\eta$ in Eq. (1a) yields

$$\dot{\beta} = \left(-\iota\delta\Omega - \frac{\Gamma}{2}\right)\beta + ig(\alpha_0^{*}\eta + \alpha_0\eta^{\dagger})e^{\iota\Omega_o t} + f(t) \quad (6)$$

where asynchronous terms are discarded. Also, $\Gamma \equiv \Gamma_o + A_- - A_+$ where $A_\pm$ are the Stokes (+) and anti-Stokes (−) rates in the notation of Ref. [25]. As is well known, the sign of $A_- - A_+$ is set by the pump detuning ($\Delta\omega$) such that damping (cooling) results for $\Delta\omega<0$ while negative damping (gain and/or amplification) results for $\Delta\omega>0$. The term $\delta\Omega$ is associated with the optical spring.

For comparison with recent work [25,26], the case of a red-detuned pump is considered first (cooling). The time evolution of $n \equiv \langle \beta^\dagger \beta\rangle$ is calculated using Eq. (6),

$$\dot{n} = -\Gamma(n-\bar{n}), \quad (7a)$$

$$\bar{n} = \frac{\Gamma_o}{\Gamma} n_T + \frac{A_+(N_{T'}+1) + A_- N_{T'}}{\Gamma}, \quad (7b)$$

where n_T ($N_{T'}$) is the thermal occupation of the mechanical (optical) bath at temperature T (T') and frequency Ω_o (ω_o). For $N_{T'}$ negligible (optical pumping), this result is identical to those of previously reported work [25,26]. The terms involving the optical thermal occupation $N_{T'}$ are new and imply that some cooling of the pump fields themselves is required in cases of microwave pumping [31].

For a blue-detuned pump the sign of $A_- - A_+$ is negative (antidamping or gain) and $G_o \equiv A_+ - A_-$ is the unsaturated mechanical gain. Since both A_+ and A_- depend linearly on the optical pumping (see below), $\Gamma=\Gamma_o - G_o$ in Eq. (6) is reduced with increased optical pumping. The threshold level of pumping is set by the condition $G_o=\Gamma_o$. Beyond this level of pumping the mechanical motion exhibits self-sustained oscillations as first reported in [19–21]. It is straightforward to show that higher-order terms in the perturbation expansion induce saturation of the circulating pump field, and thereby "clamp" the mechanical gain to its threshold value for pumping above threshold. Physically, this saturation occurs as circulating pump power is scattered into the motional sidebands; and, to leading order, the saturated mechanical gain takes the form $G=G_o-\kappa\bar{n}_c$ where κ is a saturation parameter that depends on pump detuning. This behavior will be detailed elsewhere. In this *above-threshold* regime, the motion can be approximated as $\beta(t)=\sqrt{\bar{n}_c}[1+\rho(t)]\exp[\iota\phi(t)]\exp(-\iota\delta\Omega t)$, where $\rho(t)$ and $\phi(t)$ are amplitude and phase fluctuation operators. Using this form, Eq. (6) is decomposed into equations of motion for the small-signal amplitude and phase as in [32]. The frequency-fluctuation spectral density $W_{\dot{\phi}}(\Omega)$ [32] of the operator $\dot{\phi}(t)$ is determined accordingly as

$$W_{\dot{\phi}}(\Omega) = \frac{(A_+^- + A_-^-)(N_{T'}+1) + (A_+^+ + A_-^+)N_{T'}}{4\bar{n}_c} + \frac{\Gamma_o}{4\bar{n}_c}\left[1 + \left(1+\frac{\Omega}{\Omega_o}\right)n_T(\Omega_o+\Omega) + \left(1-\frac{\Omega}{\Omega_o}\right)n_T(\Omega_o-\Omega)\right], \quad (8a)$$

where

$$A_\pm^+(\Omega) \equiv g^2|\alpha_o|^2\left(\frac{\gamma_o}{[\Omega+(\Delta\omega\mp\Omega_o)]^2+\gamma_o^2/4}\right), \quad (8b)$$

$$A_\pm^-(\Omega) \equiv g^2|\alpha_o|^2\left(\frac{\gamma_o}{[\Omega-(\Delta\omega\mp\Omega_o)]^2+\gamma_o^2/4}\right). \quad (8c)$$

An equivalent way to represent the phase noise would be through the phase noise spectral density function given by $W_\phi(\Omega)=W_{\dot{\phi}}(\Omega)/\Omega^2$. Also, the quantities in Eqs. (8b) and (8c) are related to the Stokes and anti-Stokes rates as follows: $A_\pm = A_\pm^+(\Omega=0)=A_\pm^-(\Omega=0)$.

Since $W_{\dot{\phi}}(\Omega)$ is spectrally flat in the vicinity of the origin, the fundamental line shape of the oscillator is approximately Lorentzian with a linewidth given by $\Delta\Omega = W_{\dot{\phi}}(0)$. Upon simplification this yields

$$\Delta\Omega = \frac{[\Gamma_o n_T + A_+ + (A_+ + A_-)N_{T'}]}{2\bar{n}_c}, \quad (9a)$$

$$\Delta\Omega|_{\text{optical pump}} = \frac{\Gamma_o n_T}{2\bar{n}_c} + \frac{A_+}{2\bar{n}_c}, \tag{9b}$$

where the form in Eq. (9b) applies for the case of an optical pump ($N_{T'} \approx 0$) and where the threshold condition $\Gamma_o = A_+ - A_-$ has been used to simplify these expressions. Equation (9b) is the key result of this analysis. The first term in Eq. (9b) is the classical contribution to the oscillator linewidth. It dominates at room temperature and has recently been measured in a microtoroidal oscillator [23,24]. The second term is new, giving the phase noise from quantum back action.

To estimate this back-action component, consider a case where $A_+ \gg A_-$ which is typical of operation in the sideband-resolved (SR) regime. The threshold condition reduces to $\Gamma_o = A_+$ and the linewidth formula takes on the simplified form

$$\Delta\Omega|_{\text{SR}} = \frac{\Gamma_o}{2\bar{n}_c}(n_T + 1). \tag{10}$$

In [23,24] the phase noise of a regenerative oscillator at 54.2 MHz was studied. The unpumped mechanical Q of that oscillator was 2000 and regenerative linewidths narrower than 1 Hz were observed at room temperature ($n_T \approx 116\,000$). These results were in good agreement with the classical portion of Eq. (9b), in terms of both the amplitude dependence and estimated operational temperature. Therefore, using Eq. (10) for estimation purposes, the back-action component of the linewidth is expected to be n_T times smaller than 1 Hz (tens of microhertz) for this device. (As an aside, $n_o \approx 2\times 10^{10}$ and $\bar{n}_c \approx 2\times 10^{9}$ at a linewidth of 1 Hz, so that the operational range of the device in [23,24] falls within the regime $R<1$ where second-order sidebands remain weaker than first-order sidebands.) Higher mechanical Q's would improve this performance [33]. On the other hand, both the magnitude of the back-action noise and the required operational temperature to observe it can be boosted by oscillating on higher-frequency, lower-mechanical Q modes. The device in Ref. [22] should present observable back-action phase noise at $T=50$ mK, a temperature obtainable using dilution refrigeration. Boosts to eigenfrequencies enabled by silicon photonics [7] would provide further increases to the operational temperature.

It is interesting to note that a correction to the linewidth is caused by the optical spring effect. The corrected linewidth $\Delta\Omega_C$ takes on the form

$$\Delta\Omega_C = \Delta\Omega(1+\epsilon^2), \tag{11a}$$

$$\epsilon \equiv \frac{(\Delta\omega - \Omega_o)A_+ + (\Delta\omega + \Omega_o)A_-}{(\gamma_o/2)(A_+ - A_-)}, \tag{11b}$$

where $\Delta\Omega$ is given by Eq. (9a). The contribution is small in cases where $\Omega_o \gg \gamma_o/2$, which is typical for linewidth studies performed to date. This correction is created by the power dependence of the spring term $\delta\Omega$ in Eq. (6). Back action and thermal noise can couple though this term as a result of it dependence upon power. The effect is well known in detuned laser oscillators.

Before concluding, the Langevin normalizations are given. Since the bath temperature can, in principle, be very low, it is important to consider the effect of temperature on the time correlation behavior of the Langevin operators. A damped oscillator subject to coupling with a bath of oscillators at temperature T obeys the following quantum Langevin equations [29]:

$$\dot{x} = \frac{p}{m}, \tag{12a}$$

$$\dot{p} = -m\Omega_o^2 p - \Gamma_o p + \xi(t), \tag{12b}$$

where commutators and time correlations for the Langevin operator $\xi(t)$ are given in [29]. Equation (1a) (without radiation pressure coupling) follows from these equations by defining raising and lowering operators in the standard way and dropping asynchronous terms under the rotating wave approximation. This is justified as it is assumed that $\Gamma_o \ll \Omega_o$. The Langevin operators $f(t)$ and $f^\dagger(t)$ are then related to $\xi(t)$ as follows:

$$\xi(t) = \iota\sqrt{2m\hbar\Omega_o}[f^\dagger(t)e^{\iota\Omega_o t} - f(t)e^{-\iota\Omega_o t}], \tag{13a}$$

$$f(t) = \sqrt{\frac{\Gamma_o}{2\pi\Omega_o}}\int_0^\infty d\omega\sqrt{\omega}\alpha(\omega,t), \tag{13b}$$

where $\alpha(\omega,t)$ is a lowering operator for a bath oscillator having frequency ω. For a thermal bath, the time correlations for $f(t)$ and $f^\dagger(t)$ are given by

$$\langle f^\dagger(t+\tau)f(t)\rangle = \frac{\Gamma_o}{2\pi\Omega_o}\int_0^\infty d\omega\ \omega n_T e^{\iota(\omega-\Omega_o)\tau}, \tag{14a}$$

$$\langle f(t+\tau)f^\dagger(t)\rangle = \frac{\Gamma_o}{2\pi\Omega_o}\int_0^\infty d\omega\ \omega(n_T+1)e^{-\iota(\omega-\Omega_o)\tau}, \tag{14b}$$

where $n_T(\omega)$ is the thermal occupancy at temperature T of a bath oscillator having frequency ω. Lower temperatures therefore create increasingly longer correlation times. While the analysis used to compute the oscillator linewidth [Eqs. (9a) and (9b)] and the frequency-fluctuation spectral density [see Eq. (8a)] uses this general form for the time correlations of the Langevin operator $f(t)$, it is nonetheless possible (and easier) to approximate the time correlations as δ functions provided that the spectral width of the oscillator is substantially narrower than the width of the thermal distribution function, $n_T(\omega)$ (i.e., $\hbar\Gamma_o \ll k_B T$). In effect, the Langevin sources are effectively "white" noise in comparison to the oscillator dynamics when this condition is satisfied. Concerning the operator $\ell(t)$ for the optical mode, the corresponding bath is at optical frequencies and its temperature T' can be much higher than the mechanical oscillator with no adverse effect on calculated linewidth. Accordingly, the correlation functions for $\ell(t)$ are taken as $\langle \ell^\dagger(t+\tau)\ell(t)\rangle = \gamma_o N_{T'}\delta(\tau)$ and $\langle \ell(t+\tau)\ell^\dagger(t)\rangle = \gamma_o(N_{T'}+1)\delta(\tau)$.

In summary, the back-action limit to the linewidth in an optomechanical oscillator has been derived. Technical noise limits have not been addressed in this analysis. However, as these oscillators are optically pumped, they feature a high degree of isolation from electrical and electromagnetic interference, and can leverage quantum-limited sources. Moreover, technical noise in these systems is being systematically addressed as part of current efforts directed toward realization of ground-state cooling. All of these considerations bode well for future improvements in oscillator stability. Finally, the back-action limit to the linewidth is one that is interesting purely from a physical viewpoint. Not only can it be enhanced by design, potentially enabling observation of quantum-back-action noise in the macroscopic realm, but, despite different underlying physics, it has a similarity in form to the Schawlow-Townes formula [34] for laser oscillators.

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